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Combinatorial Counterpart of Some q −Series in the light of Colour Partitions

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Abstract

The n −Colour partition, introduced by Agarwal and Andrews [1] are extended to $(n + t)$–color partitions where the parts of size n can come in $(n + t)$, $t \geq 0$, different colors. In the study of partition theory, the q −series plays an important role. The q −series are integral part of partition identities and hence for deep combinatorial study of analytic partition identities, it is important to know the combinatorial counterpart of q −series first.

In this paper, the combinatorial counterpart of some analytic q −series has been given in the light of Colour Partitions

Keywords: q −series, n −Colour partition, $(n + t)$–color partitions, Generating functions, Partition functions, Combinatorial interpretations, etc.

1. Introduction:

Throughout this paper, we assume $|q|<1$and, as customary, we define

$$(a;q)_0 = 1$$
$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \text{ for } n\geq 1$$

and,

$$(a;q)_\infty = \prod_{k=1}^{\infty}(1-aq^k).$$

It follows that $(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$

The multiple q-shifted factorial is defined by

$$(a_1, a_2, \ldots\ldots a_m; q)_n = (a_1;q)_n (a_2;q)_n \ldots . (a_m;q)_n$$
$$(a_1, a_2, \ldots\ldots a_m; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \ldots . (a_m;q)_\infty$$

Definition 1: An n- colour partition (also called a partition with n copies of n) of a positive integer μ is a partition in which a part of size n, $(n \geq 0)$ can come in n different colours denoted by the subscripts: n_1, n_2, n_3, ..., n_n and the parts satisfy the order:

$$1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 3_3 < 4_1 < 4_2 < 4_3 < 4_4 < 5_1 < \cdots \ldots \ldots$$

For example, the six n colour partitions of 3 are:

$$3_1,\ \ 3_2,\ \ 3_3,\ \ 2_1+1_1,\ \ 2_{2+}1_1,\ \ 1_1+1_1+1_1$$

The concept of $(n+t)$- colour partitions or "$(n+t)$ copies of n", $t \geq 0$ is an extended work of n-colour partition.

Definition 2: A partition with "$(n+t)$ copies of n", $t \geq 0$, is a partition in which a part of size $n, n \geq 0$, can come in $n+t$ different colours denoted by subscripts n_1, n_2, n_3, ..., n_{n+t}. The parts in an $(n+t)$- colour partition can be arranged lexicographically as:

$$1_1 < 1_2 < 1_3 < 2_1 < 2_2 < 2_3 < 3_1 < 3_2 < 3_3 < \cdots \ldots \ldots ..$$

Note that zero appears as a part if $t \geq 1$ and also zeros are not allowed to repeat in any partition.

For example, there are twenty $(n+2)$- colour partitions of 2 as follows:

2_1	$2_1 0_2$	$1_1 1_1$	$1_3 1_2$	$1_3 1_1 0_2$
2_2	$2_2 0_2$	$1_2 1_1$	$1_3 1_3$	$1_2 1_2 0_2$
2_3	$2_3 0_2$	$1_3 1_1$	$1_1 1_1 0_2$	$1_3 1_2 0_2$
2_4	$2_4 0_2$	$1_2 1_2$	$1_2 1_1 0_2$	$1_3 1_3 0_2$

If $m_i, n_j, m \geq n$ are any two parts of an n colour partition, then their weighted difference is defined by $m - n - i - j$ and is denoted by $((m_i - n_j))$.

Since the $(n+t)$- colour partitions are only the extensions of n- colour partitions, so the weighted difference among any two parts m_i, n_j, in an $(n+t)$- colour partitions is same as defined n- colour partitions.

If $P(\mu)$denote the number of n-colour partitions of γ then the generating function $F(q)$ for $P(\mu)$ is given by,

$$F(q) = \sum_{\gamma=0}^{\infty} P(\mu) q^{\mu} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}$$

(1.1)

Using the notion of n-colour partition, Agarwal [4] obtained the combinatorial interpretation of the following generalised basic q −series

$$\sum_{\gamma=0}^{\infty} B_k(\mu) q^{\mu} = \frac{q^{n\left(1+\frac{(k+3)(n-1)}{2}\right)}}{(q;q)_n (q;q^2)_n} \tag{1.2}$$

where $(q;q)_n = \prod_{k=1}^{n} (1-q^k)$ and $(q;q)_{\infty} = \prod_{k=1}^{\infty} (1-q^k)$for |q|<1, as follows:

Theorem 1.1: For $k \geq -3$, $B_k(\mu)$ represents the number of n-colour partitions of μ such that each pair of parts m_i, n_j satisfies $((m_i - n_j)) > k$.

For $k = 0, -1, -2$, Theorem 1.1, in view of the identities viz,

$$\sum_{n=0}^{\infty} \frac{q^{n(3n-1)/2}}{(q;q)_n (q;q^2)_n} = \frac{(q^4, q^6, q^{10}; q^{10})_{\infty}}{(q;q)_{\infty}} \tag{1.3}$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n(q;q^2)_n}=\frac{(q^6,q^8,q^{14};q^{14})_\infty}{(q;q)_\infty} \tag{1.4}$$

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q;q)_n(q;q^2)_n}=\frac{(-q;q)_\infty}{(q;q)_\infty}\frac{(q^2,q^5,q^7;q^7)_\infty}{(-q,-q^6;q^7)_\infty} \tag{1.5}$$

from [6, Eqn. (46), (61)] and [7, Eqn. (3.1)] reduces to the following Theorems (1.2), (1.3) and (1.4).

Theorem (1.2): The number of n −colour partitions of μ such that each pair of parts m_i, n_j, $m_i \geq n_j$ satisfies the weighted difference $((m_i - n_j)) > 0$ is equal to the number of ordinary partitions of μ into parts $\not\equiv 0, \pm 4$ (mod 10).

Theorem (1.3): The number of n −colour partitions of μ such that each pair of parts m_i, n_j, $m_i \geq n_j$ satisfies the weighted difference $((m_i - n_j)) > 0$ is equal to the number of ordinary partitions of μ into parts $\not\equiv 0, \pm 6$ (mod 14).

Theorem (1.4): The number of n −colour partitions of μ such that each pair of parts m_i, n_j, $m_i \geq n_j$ satisfies the weighted difference $((m_i - n_j)) > 0$ is equal $\sum_{k=0}^{\mu} C(\mu - k)D(k)$, where $C(\mu)$ denote the number of partitions of μ into distinct parts $\equiv \pm 3$ (mod 7) and $D(\mu)$ denote the number of partitions of μ into distinct parts $\equiv \pm 4$ (mod 14).

Recently, A.K. Agarwal and M. Rana [8], obtains the n −colour partition theoretic interpretation of a generalised q −series as follows:

Theorem 1.5: For a given positive integer k, let $G_k(\mu)$ represent the n −colour partitions of μ into parts greater than or equal to k such that first copy(resp. second copy) of the odd parts (resp. even parts) and the second copy (resp. first copy) of the even parts (resp. odd parts) appear if k is odd (resp. even). The weighted difference between any two parts is nonnegative and even. Then

$$\sum_{\gamma=0}^{\infty} G_k(\mu)q^{\mu} = \frac{(-q;q^2)_n q^{n(n+k-1)}}{(q^2;q^2)_n}$$

The Theorem 1.5 facilitates to give the n −colour combinatorial interpretations of the famous analytic versions of Gollnitz-Gordon identities as listed in [6, I(36), I(34)] and [9, Cor. 2.7, p.21 with $q = q^2$ and $a = -2$] for the sum sides.

2. Main Results:

In this section, we give the combinatorial interpretations of some $q-$ Series with the help of $(n+t)-$ colour partitions:

Theorem 2.1: If $P_1(\mu)$ denote the number of partitions of μ with n copies of n into parts greater than or equal to 2 such that if m_i is the least or only part in the partition then $m - i \equiv 2\ (mod\ 4)$ and the weighted difference between consecutive parts is non-negative and $\equiv 0\ (mod\ 4)$. Then the generating function of $P_1(\mu)$ is given by

$$\sum_{n=0}^{\infty} P_1(\mu)\, q^{\mu} = \sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q^4;q^4)_n(q;q^2)_n}$$

Example 2.1: For μ =8, there are three partitions enumerated by $P_1(8)$ are:

$8_2, 8_6$ and $5_1 3_1$

Proof of Theorem 2.1: Let $P_1(\mu, m)$ denote the number of partitions enumerated by $P_1(\mu)$into exactly m parts.We split the partitions enumerated by $P_1(\mu, m)$ into the following three classes:

(i) those that do not contain k_{k-1} as a part.
(ii) those that contain 2_1 as a part.
(iii) those that contain $k_{k-2}, (k > 2)$ as a part.

We now transform the partitions in class (i) by subtracting 4 from each part ignoring the subscripts, it will not disturb the inequalities between the parts and so the transformed partition will be of the type enumerated by $P_1(m, \mu - 4m)$.

Next, we transform the partitions in class (ii) by deleting the part $2_1$and then subtracting 4 from all the remaining parts ignoring the subscripts. The transformed partition will be of the type enumerated by $P_1(m-1, \mu - 4m + 2)$.

Finally, we transform the partitions in class (iii) by replacing k_{k-2} by $(k+1)_{(k-3)}$ and then subtracting 2 from all the remaining parts. This will produce a partition of $\mu - 2m$ +1 into m parts. It is important to note that, by this transformation we will get only the partitions of $\mu - 2m + 1$ into m parts which contain a part of the form k_{k-2} . Therefore the actual number of partitions which belongs to class (iii) is $P_1(m, \mu - 2m + 1) - P_1(m, \mu - 6m + 1)$, where $P_1(m, \mu - 6m + 1)$ is the number of partitions of $\mu - 2m + 1$ into m parts which are free from parts like k_k or $k_{(k-2).}$

The above transformations are reversible and hence establish bisection between the partitions enumerated by $A_1(m, \gamma)$ and those enumerated by

$$P_1(m, \mu - 4m) + P_1(m-1, \mu - 4m + 2) + P_1(m, \mu - 2m + 1) - P_1(m, \mu - 6m + 1)$$

This leads to the recurrence relation

$$P_1(m, \mu) = P_1(m, \mu - 4m) + P_1(m-1, \mu - 4m + 2) + P_1(m, \mu - 2m + 1) - P_1(m, \mu - 6m + 1) \quad (2.1)$$

Let

$$g_1(z; q) = \sum_{\gamma=0}^{\infty} \sum_{m=0}^{\infty} P_1(m, \gamma) z^m q^\gamma \quad (2.2)$$

Substituting $P_1(m, \gamma)$ from (2.1) into (2.2), we get,

$$g_1(z; q) = g_1(zq^4; q) + zq^2\, g_1(zq^4; q) + q^{-1} g_1(zq^2; q) - q^{-1} g_1(zq^6; q) \quad (2.3)$$

Consider

$$g_1(z; q) = \sum_{n=0}^{\infty} \alpha_n(q) z^n \quad (2.4)$$

Now, using (2.4) in (2.3) and then comparing the coefficients of z^n , we get

$$\alpha_n(q)= q^{4n}\alpha_n(q) + q^{4n-2}\alpha_{n-1}(q) + q^{2n-1}\alpha_n(q) - q^{6n-1}\alpha_n(q) \quad (2.5)$$

Hence,

$$\alpha_n(q)= \frac{q^{4n-2}}{(1-q^{4n})(1-q^{2n-1})}\,\alpha_{n-1}(q) \quad (2.6)$$

Iterating (2.6) n −times and noting that $\alpha_0(q) = 1$, we get,

$$\alpha_n(q) = \frac{q^{2n^2}}{(q^4;q^4)_n(q;q^2)_n} \quad (2.7)$$

Putting the value of $\alpha_n(q)$ in (2.4), we get

$$g_1(z;q) = \sum_{n=0}^{\infty} \alpha_n(q) z^n$$

$$g_1(z;q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^4;q^4)_n(q;q^2)_n} . z^n$$

$$g_1(1;q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^4;q^4)_n(q;q^2)_n}$$

Now, since

$$\sum_{\gamma=0}^{\infty} P_1(\mu) q^{\gamma} = \sum_{\mu=0}^{\infty}\left(\sum_{m=0}^{\infty} P_1(m,\mu)\right) q^{\mu}$$

$$= g_1(1;q)$$

$$= \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^4;q^4)_n(q;q^2)_n} \quad (2.8)$$

This completes the proof.

Theorem 2.2. let $P_2(\mu)$ denote the number of partitions of μ with "n *copies of* n" into parts greater than or equal to 3 such that if m_i is the least or the only part in the partition then $m - i \equiv 2 (\text{mod} 4)$ and weighted difference between consecutive parts is non-negative and $\equiv 0 (\text{mod} 4)$

Then the generating function of $P_2(\mu)$ is given by

$$\sum_{n=0}^{\infty} P_2(\mu)\, q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q^4;q^4)_n(q;q^2)_n}$$

Proof: let $P_2(m,\mu)$ denote the number of partitions of μ enumerated by $P_2(\mu)$ into m parts. We split the partitions enumerated

by $P_2(m,\mu)$ into three classes:

(i) those that do not contain k_k-2 as a part,

(ii) those that contain 3_1 as a part,

(iii) those that contain $k_{k-2} (k > 3)$ as a part.

We now transform the partitions into class (i) by subtracting 4 from each part ignoring the subscripts, it will not disturb the inequalities between the parts and transformed partition will be of the type enumerated by $P_2(m, \mu - 4m)$. Next, transform the partitions in class (ii) by deleting the least part 3_1 and then subtracting 2 from all the remaining parts ignoring the subscripts. The transformed partition will be of the type enumerated by $P_2(m-1, \mu - 2m - 1)$. Finally, we transform the partitions in class (iii) by replacing the part k_{k-2} by $(k+1)_{k-3}$ and then subtracting 2 from all the parts. This will produce the partitions of (μ− 2m + 1) into m parts. Note here that, by this transformation we will get only those partitions of (μ− 2m + 1) into m parts which contain a part of the form k_{k-2}.

Therefore, the actual number of partitions which belong to class (iii) is $P_2(m,\mu-2m+1)-P_2(m,\mu-6m+1)$ where $P_2(m,\mu-6m+1)$ is the number of partitions of $\mu-2m+1$ into m parts which are free from parts like K_{k-2}.

The above transformations are clearly reversible and bijection between the partitions enumerated by $P_2(m,\mu)$ and those enumerated by $P_2(m,\mu-4m)$ $+P_2(m-1,\mu-2m-1)$ $+P_2(m-1,\mu-2m+1)$ $-P_2(m,\mu-6m+1)$

This leads to the recurrence relation

$$P_2(m,\mu)=P_2(m,\mu-4m)+P_2(m-1,\mu-2m-1)+P_2(m-1,\mu-2m+1)-P_2(m,\mu-6m+1) \tag{2.9}$$

Now let,

$$g_2(z;q)=\sum_{\mu=0}^{\infty}\sum_{m=0}^{\infty}P_2(m,\mu)z^m q^\mu \tag{2.10}$$

Substituting $P_2(m,\ \mu)$ from (2.9) into (2.10), we get,

$$g_2(z;q)=g_2(zq^4;q)+zq^3\,g_2(zq^2;q)+q^{-1}g_2(zq^2;q)-q^{-1}g_2(zq^6;q) \tag{2.11}$$

Consider

$$g_2(z;q)=\sum_{n=0}^{\infty}\beta_n(q)z^n \tag{2.12}$$

Now, using (2.12) in (2.11) and then comparing the coefficients of z^n , we get

$$\beta_n(q)=q^{4n}\beta_n(q)+q^{2n+1}\beta_{n-1}(q)+q^{2n-1}\beta_n(q)-q^{6n-1}\beta_n(\mathrm{q})$$

Hence,

$$\beta_n(q)=\frac{q^{2n+1}}{(1-q^{4n})(1-q^{2n-1})}\beta_{n-1}(q) \tag{2.13}$$

Iterating (2.13) n −times and noting that $\beta_0(q)=1$, we get,

$$\beta_n(q)=\frac{q^{n^2+2n}}{(q^4;q^4)_n(q;q^2)_n} \tag{2.14}$$

Putting the value of $\alpha_n(q)$ in (2.12), we get

$$g_2(z;q)=\sum_{n=0}^{\infty}\beta_n(q)z^n$$

$$g_2(z;q)=\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q^4;q^4)_n(q;q^2)_n}.z^n$$

$$g_2(1;q)=\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q^4;q^4)_n(q;q^2)_n}$$

Now,

$$\sum_{\gamma=0}^{\infty}P_2(\mu)q^\mu=\sum_{\mu=0}^{\infty}\left(\sum_{m=0}^{\infty}P_2(m,\mu)\right)q^\mu$$

$=\sum_{\mu,m=0}^{\infty} P_2(m,\mu)\, q^{\mu}$
$= g_2(1;q)$
$= \sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^4;q^4)_n(q;q^2)_n}$

This completes the proof.

Theorem 2.3. let $P_3(\mu)$ denote the number of partitions of μ with "n *copies of* n" into parts greater than or equal to 3 such that if m_i is the least or the only part in the partition then $m \equiv$i(mod4) and weighted difference between consecutive parts is non-negative and $\equiv$ 0(mod4)

Then the generating function of $P_3(\mu)$ is given by

$$\sum_{n=0}^{\infty} P_3(\mu)\, q^{\mu} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n(q;q^2)_n} \tag{2.15}$$

Proof: let $P_3(m,\mu)$ denote the number of partitions of μ enumerated by $P_3(\mu)$ into m parts. We split the partitions enumerated

by $P_3(m,\mu)$ into three classes:

(i) those that do not contain k_k as a part,

(ii) those that contain 1_1 as a part,

(iii) those that contain k_k (k > 2) as a part.

We now transform the partitions into class (i) by subtracting 4 from each part ignoring the subscripts, it will not disturb the inequalities between the parts and transformed partition will be of the type enumerated by $P_3(m, \mu - 4m)$.

Next, transform the partitions in class (ii) by deleting the least part 1_1 and then subtracting 2 from all the remaining parts ignoring the subscripts. The transformed partition will be of the type enumerated by $P_3(m-1, \mu - 2m + 1)$.

Finally, we transform the partitions in class (iii) by replacing the part k_k by $(k+1)_{k-1}$ and then subtracting 2 from all the parts. This will produce the partitions of (μ− 2m + 1) into m parts. Note here that, by this transformation we will get only those partitions of (μ− 2m + 1) into m parts which contain a part of the form k_k.

Therefore, the actual number of partitions which belong to class (iii) is $P_3(m,\mu - 2m + 1) - P_3(m,\mu - 6m + 1)$ where $P_3(m,\mu - 6m + 1)$. is the number of partitions of $\mu - 2m + 1$ into m parts which are free from parts like K_{k-2}.The above transformations are clearly reversible and bijection between the partitions enumerated by $P_3(m,\mu)$ and those enumerated by $P_3(m,\mu - 4m)$ $+P_3(m-1,\mu - 2m + 1)$ $+P_3(m,\mu - 2m + 1)$ $-P_3(m,\mu - 6m + 1)$

This leads to the recurrence relation

$$P_3(m,\mu) = P_3(m,\mu - 4m) + P_3(m-1,\mu - 2m + 1) + P_3(m,\mu - 2m + 1) - P_3(m,\mu - 6m + 1) \tag{2.16}$$

Now let,

$$g_3(z;q) = \sum_{\mu=0}^{\infty} \sum_{m=0}^{\infty} P_3(m,\mu) z^m q^{\mu} \tag{2.17}$$

Proceeding as same as in proof of Theorem 2.2, we get the following q–functional equation

$$g_3(z;q) = g_3(zq^4;q) + zq\, g_3(zq^2;q) + q^{-1} g_3(zq^2;q) - q^{-1} g_3(zq^6;q) \tag{2.18}$$

Consider

$$g_3(z;q) = \sum_{n=0}^{\infty} \gamma_n(q) z^n \tag{2.19}$$

Now, since $g_3(0;q) = 1$, using (2.19) in (2.18) and then comparing the coefficients of z^n , we get

$\gamma_n(q)= q^{4n}\gamma_n(q) + q^{2n-1}\gamma_{n-1}(q) + q^{2n-1}\gamma_n(q) - q^{6n-1}\gamma_n(q)$

Hence,

$$\gamma_n(q)= \frac{q^{2n-1}}{(1-q^{4n})(1-q^{2n-1})}\gamma_{n-1}(q) \tag{2.20}$$

Iterating (2.20) n −times and noting that $\gamma_0(q) = 1$, we get,

$$\gamma_n(q) = \frac{q^{n^2}}{(q^4;q^4)_n(q;q^2)_n} \tag{2.21}$$

Putting the value of $\gamma_n(q)=$ in (2.10), we get

$$g_3(z;q) = \sum_{n=0}^{\infty} \gamma_n(q) z^n$$

$$g_3(z;q) = \sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^4;q^4)_n(q;q^2)_n}.z^n$$

$$g_3(1;q) = \sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^4;q^4)_n(q;q^2)_n}$$

Now,

$$\sum_{\gamma=0}^{\infty} P_3(\mu)q^{\mu} = \sum_{\mu=0}^{\infty}\left(\sum_{m=0}^{\infty} P_3(m,\mu)\right)q^{\mu}$$

$$= \sum_{\mu,m=0}^{\infty} P_3(m,\mu)\, q^{\mu}$$

$$= g_3(1;q)$$

$$= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n(q;q^2)_n}$$

This completes the proof.

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