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# Approximate and Weak Separation in Cartesian Products of Function Algebras

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## Abstract
Classical separation properties, including regularity and normality, have been widely studied within the framework of commutative Banach algebras and function algebras. In this paper, we extend these investigations to Cartesian products of function algebras by considering their generalized forms: approximate regularity, approximate normality, and weak regularity. We examine the interrelations among these properties and establish conditions characterizing their validity in the product setting.

**Keywords:** Approximate regularity, Approximate normality, Weak regularity

## 1. Introduction
Various separation axioms are well studied in Topology. For a function algebra on a compact Hausdorff space $X_1$, the space $X_1$ enjoys strong separation properties mainly due to the Urysohn's lemma. These topological properties extend naturally to function algebras and, more generally, to commutative Banach algebras. If $A_1$ is a function algebra on $X_1$, it may also be regarded as a function algebra on $\Delta(A_1)$, its maximal ideal space. Thus, these properties can be considered on either $X_1$ or $\Delta(A_1)$.

Weaker analogues of these notions, namely approximate regularity, approximate normality, and weak regularity, have also been introduced to extend the separation framework to function algebras. Here we investigate these properties for the Cartesian product of function algebras.

Let $A_1$ and $A_2$ be function algebras on compact Hausdorff spaces $X_1$ and $X_2$ respectively. Then with coordinate wise operations $A_1 \times A_2$ is a function algebra on the topological sum $X_1 + X_2$ with respect to the norm $\|(f_1, f_2)\| = \max\{\|f_1\|_\infty, \|f_2\|_\infty\}$, where $f_1 \in A_1$, $f_2 \in A_2$ [8]. Note that $X_1 + X_2$ is the topological space $X_1 \cup X_2$ with the sum topology. Nirav Shah et. al. establishes that $\Delta(A_1 \times A_2) = \Delta(A_1) + \Delta(A_2)$ with the Gelfand topology, where $\Delta(A_1)$ denotes the maximal ideal space of $A_1$ [11]. Note that if $\varphi \in \Delta(A_1 \times A_2)$, then $\varphi = \varphi_{A_1} \in \Delta(A_1)$ or $\varphi = \varphi_{A_2} \in \Delta(A_2)$ where $\varphi_{A_1}$ and $\varphi_{A_2}$ are defined as $\varphi_{A_1}(f_1) = \varphi(f_1, 0), \forall f_1 \in A_1$ and $\varphi_{A_2}(f_2) = \varphi(f_2, 0), \forall f_2 \in A_2$ . On the other hand, if $\varphi_{A_1} \in \Delta(A_1)$ $\left(\varphi_{A_2} \in \Delta(A_2)\right)$, then $\varphi \in \Delta(A_1 \times A_2)$ is defined as $\varphi(f_1, f_2) = \varphi_{A_1}(f_1)$ $(\varphi(f_1, f_2) = \varphi_{A_2}(f_2))$ , $\forall (f_1, f_2) \in A_1 \times A_2$. The Cartesian product of regular commutative Banach algebras with identity is examined in [13].

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Throughout the paper, unless specified otherwise, $A_1$ and $A_2$ will be regarded as function algebras on compact Hausdorff spaces $X_1$ and $X_2$ respectively.

## 2. Approximate Separation: Regularity and Normality

This section is devoted to the study of approximate regularity and approximate normality for the Cartesian products of function algebras. The concepts were first introduced by Wilken [14].

**Definitions 2.1.** Let $A_1$ be a function algebra on $X_1$. Then
(1) $A_1$ is said to be approximately regular on $\Delta(A_1)$, if for each closed subset E of $\Delta(A_1)$, $\varphi \in \Delta(A_1) \setminus E$ and $\epsilon > 0$, there exists $f_1 \in A_1$ such that $\widehat{f_1}(\varphi) = 1$ and $\|\widehat{f_1}\|_E < \epsilon$.
(2) $A_1$ is said to be approximately normal on $\Delta(A_1)$, if for each disjoint pair of closed subsets $E_1$ and $E_2$ of $\Delta(A_1)$, and $\epsilon > 0$, there exists $f_1 \in A_1$ such that $\|\widehat{f_1}\|_{E_1} < \epsilon$ and $\|1 - \widehat{f_1}\|_{E_2} < \epsilon$.

Both of the above notions can equally be formulated on $X_1$.

**Remarks 2.2**. (1) It is evident that regularity entails approximate regularity, and normality entails approximate normality; however, the converse of these implications does not hold. [ Example 2.4 (2)].
(2) For a function algebra $A_1$, approximate normality and approximate regularity are equivalent on $\Delta(A_1)$, [14]. But this equivalence does not hold when considered on $X_1$. [Example 2.4 (3)].

Following are some interesting results about these separation axioms [7, 14].
**Properties 2.3.** (1) If $\mathrm{ch}(A_1) = X_1$, then $A_1$ is approximately regular on $X_1$, where $\mathrm{ch}(A_1)$ is the Choquet boundary of $A_1$. Hence every URM (unique representing measure) algebra [9] is approximately regular.
(2) Let $A_1$ be approximately regular or approximately normal function algebra on $X_1$. Then $\Gamma(A_1) = X_1$, where $\Gamma(A_1)$ is the Šilov boundary of $A_1$.
(3) If $A_1$ is maximal on $X_1$, then $A_1$ is approximately regular on $X_1$.
(4) Every Dirichlet algebra is approximately normal. However, logmodular algebra may not be approximately normal [Example 2.4 (3)].
(5) If $X_1$ is a compact totally disconnected space, then $C(X_1)$ is the only approximately normal function algebra on $X_1$. But the result is not true for approximately regular algebras [Example 2.4 (3)].
(6) Let $A_1$ be an approximately normal function algebra on $X_1$ and let $\{C_\alpha\}$ be the family of components of $X_1$. Then if $f_1 \in C(X_1)$ and $f_{1|C_\alpha} \in A_{1|C_\alpha}$, $\forall\alpha$, then $f_1 \in A_1$.

**Examples 2.4.** (1) $A_1 = A(D)$, the disk algebra on the unit disk D, has none of the above separation properties, as $\Gamma(A_1) = \Gamma \subsetneq D = X_1$, where Γ is the unit circle.
(2) Consider the disk algebra on the circle Γ, A(Γ). We know that A(Γ) is a Dirichlet algebra on Γ. Hence A(Γ) is approximately normal on Γ. Also A(Γ) is approximately regular on Γ as $\mathrm{ch}(A(\Gamma)) = \Gamma = X_1$. But A(Γ) is not regular and hence not normal.
(3) Let $H^\infty(m)$ be the subalgebra of $L^\infty(m)$ on the unit circle Γ consisting of all functions with negative Fourier coefficients zero. Then $H^\infty(m)$ is a logmodular algebra on $X_1 = \Delta(L^\infty(m))$ [2]. Hence it is

approximately regular. But $\Delta(L^{\infty}(m))$ being totally disconnected, $H^{\infty}(m)$ is not approximately normal on $X_1$.
(4) The standard example of fiber algebra (regular but not normal) is in fact, not approximately normal [7].

We now examine how the concepts of approximate regularity and approximate normality extend to Cartesian products of function algebras.

**Theorem 2.5.** $A_1 \times A_2$ is approximately regular on $\Delta(A_1 \times A_2)$ if and only if $A_1$ and $A_2$ are approximately regular on $\Delta(A_1)$ and $\Delta(A_2)$, respectively.

**Proof.** Suppose $A_1$ and $A_2$ are approximately regular on $\Delta(A_1)$ and $\Delta(A_2)$ respectively. Let $F$ be a closed subset of $\Delta(A_1 \times A_2)$, $\varphi \in \Delta(A_1 \times A_2) \setminus \mathrm{F}$ and $\epsilon > 0$. Since $\Delta(A_1 \times A_2) = \Delta(A_1) + \Delta(A_2)$, either $\varphi \in \Delta(A_1)$ or $\varphi \in \Delta(A_2)$. Also, for $\mathrm{F} \subset \Delta(A_1 \times A_2)$ we have three cases. First, we take $\varphi \in \Delta(A_1)$ along with all these cases. First let $\mathrm{F} \subset \Delta(A_1)$, then since $A_1$ is approximately regular, $\exists f_1 \in A_1$ such that $\widehat{f_1}(\varphi) = 1$ and $\|\widehat{f_1}\|_F < \epsilon$. Then $\mathrm{g} = (f_1, 0) \in A_1 \times A_2$ with $\hat{\mathrm{g}}(\varphi) = \widehat{f_1}(\varphi) = 1$ and $\|\hat{\mathrm{g}}\|_F = \|\widehat{f_1}\|_F < \epsilon$. Similarly, if $\mathrm{F} \cap \Delta(A_1) \neq \emptyset \neq \mathrm{F} \cap \Delta(A_2)$, then also $\mathrm{g} = (f_1, 0) \in A_1 \times A_2$, where $f_1$ is such that $\widehat{f_1}(\varphi) = 1$ and $\|\widehat{f_1}\|_{\mathrm{F} \cap \Delta(A_1)} < \epsilon$, is the required function. Finally, if $\mathrm{F} \subset \Delta(A_2)$, then the function $\mathrm{g} = (1, 0)$ of $A_1 \times A_2$ fulfills the requirements.
Similarly, we get $\mathrm{g} \in A_1 \times A_2$ satisfying the properties of approximate regularity in each case of $F$ with $\varphi \in \Delta(A_2)$, using approximate regularity of $A_2$. Thus $A_1 \times A_2$ is approximately regular on $\Delta(A_1 \times A_2)$.

Conversely, assume that $A_1 \times A_2$ is approximately regular on $\Delta(A_1 \times A_2)$. We shall show that $A_1$ is approximately regular on $\Delta(A_1)$. Let $F$ be a closed subset of $\Delta(A_1)$, $\varphi_{A_1} \in \Delta(A_1) \setminus \mathrm{F}$ and $\epsilon > 0$. Clearly, $F$ is a closed subset of $\Delta(A_1 \times A_2)$ and $\varphi_{A_1} \in \Delta(A_1 \times A_2) \setminus \mathrm{F}$. Since $A_1 \times A_2$is approximately regular, $\exists\, \mathrm{g} = (f_1, f_2) \in A_1 \times A_2$ such that $\hat{\mathrm{g}}(\varphi_{A_1}) = 1$ *and* $\|\hat{\mathrm{g}}\|_F < \epsilon$. Since $\varphi_{A_1} \in \Delta(A_1)$, we have $\hat{\mathrm{g}}(\varphi_{A_1}) = \widehat{f_1}(\varphi_{A_1})$ and as $\mathrm{F} \subset \Delta(A_1)$, $\|\hat{\mathrm{g}}\|_F = \|\widehat{f_1}\|_F$. Hence $\widehat{f_1}(\varphi_{A_1}) = 1$ and $\|\widehat{f_1}\|_F < \epsilon$. So $A_1$ is approximately regular on $\Delta(A_1)$.
Similarly, $A_2$ is approximately regular on $\Delta(A_2)$. ■

**Remarks 2.6.** (1) It follows from Remark 2.2 (2) and Theorem 2.5 that $A_1 \times A_2$ is approximately normal on $\Delta(A_1 \times A_2)$ if and only if $A_1$ and $A_2$ are approximately normal on $\Delta(A_1)$ and $\Delta(A_2)$ respectively.
(2) $A_1 \times A_2$ is approximately regular on $X_1 + X_2$ if and only if $A_1$ and $A_2$ are approximately regular on $A_1$ and $A_2$ respectively. Also, by direct method we can show that $A_1 \times A_2$ is approximately normal on $X_1 + X_2$ if and only if $A_1$ and $A_2$ are approximately normal on $X_1$ and $X_2$ respectively.

## 3. A Separation Property Weaker than Regularity

In this section, we consider a separation property weaker than regularity, known as weak regularity. The notion was introduced independently by S. J. Bhatt & H. V. Dedania [1] and by J. F. Feinstein & D. W. D. Somerset [5]. We show here that the two definitions are equivalent.

**Definitions 3.1.** Let $A_1$ be a function algebra on $X_1$ with maximal ideal space $\Delta(A_1)$. $A_1$ is said to be
(1) (S-H)-weakly regular on $\Delta(A_1)$, if given a proper closed set $E$ in $\Delta(A_1)$, there exists $f_1 \in A_1$ such that $f_1 \not\equiv 0$ and $\varphi(f_1) = 0$, for all $\varphi \in E$, i.e., $\widehat{f_1}_{|E} \equiv 0$ [1].
(2) (F)-weakly regular on $\Delta(A_1)$, if every nonempty Gelfand open subset of $\Delta(A_1)$ contains a nonempty hull-kernel open set [5].

Note that for $E \subset \Delta(A_1)$, kernel of the set $E$ is defined as $k(E) = \{f_1 \in A_1 : \widehat{f_1}_{|E} \equiv 0\}$. For an ideal I of $A_1$, the hull of ideal is defined as $h(I) = \{\varphi \in \Delta(A_1), : I \subset \ker\varphi\}$. The topology on $\Delta(A_1)$ determined by the closure operation $E \to \bar{E} = h(k(E))$, $E \subset \Delta(A_1)$, is called the hull-kernel topology on $\Delta(A_1)$ [5]. It is denoted by $\tau_{hk}^{A_1}$.

**Proposition 3.2.** A function algebra $A_1$ is (S-H)-weakly regular on $\Delta(A_1)$ if and only if it is (Fe)-weakly regular on $\Delta(A_1)$.

**Proof.** Suppose $A_1$ is (S-H)-weakly regular on $\Delta(A_1)$. Let $G$ be a proper nonempty Gelfand open subset of $\Delta(A_1)$. Consider $G^c = E$. Then $E$ is a proper Gelfand closed subset of $\Delta(A_1)$. So $\exists\, f_1 \in A_1$ such that $\widehat{f_1}_{|E} \equiv 0$ and $f_1 \not\equiv 0$. Since $f_1 \not\equiv 0$, $\widehat{f_1}(\varphi) \neq 0$, for some $\varphi \in \Delta(A_1) \setminus E = G$. Since $\widehat{f_1}_{|E} \equiv 0$, $f_1 \in k(E)$ and since $\widehat{f_1}(\varphi) \neq 0$, $\varphi \notin h(k(E))$. So $\varphi \in \left(h\left(k(E)\right)\right)^c = H$ (Say). Note that H is a Hull-kernel open subset of $\Delta(A_1)$. As $E \subset h\left(k(E)\right)$, $H \subset E^c = G$. Thus, every nonempty Gelfand open subset of $\Delta(A_1)$ contains a nonempty hull-kernel open subset of $\Delta(A_1)$. Therefore $A_1$ is (Fe)-weakly regular on $\Delta(A_1)$.

Conversely, suppose that $A_1$ is (Fe)-weakly regular on $\Delta(A_1)$. Let $F \subsetneq \Delta(A_1)$ be a Gelfand closed subset. Then $G = F^c$ is a Gelfand open subset of $\Delta(A_1)$. So there exists a nonempty hull-kernel open subset H of $\Delta(A_1)$ such that $H \subset G$. Since $H^c = E$ is hull-kernel closed set, $E = h\left(k(E)\right) \subsetneq \Delta(A_1)$. Therefore $\exists$ $\varphi \in \Delta(A_1)$ such that $\varphi \notin h(k(E))$. Therefore, there exists $f_1 \in k(E)$ such that $\varphi(f_1) = \widehat{f_1}(\varphi) \neq 0$. Thus $\widehat{f_1}_{|E} \equiv 0$ but $f_1 \not\equiv 0$. Since $F = G^c \subset H^c = E$, $\widehat{f_1}_{|F} \equiv 0$. So $A_1$ is (S-H)-weakly regular on $\Delta(A_1)$. ∎

The notions of (S–H)-weak regularity and (Fe)-weak regularity can also be formulated on $X_1$ by considering appropriate topologies. Employing arguments analogous to those used previously, one readily verifies that the two definitions are equivalent. Furthermore, the same proposition remains valid when $A_1$ is a Banach algebra.

**Remarks 3.3.** (1) If $A_1$ is a weakly regular function algebra on $\Delta(A_1)$, then $\Gamma(A_1) = \Delta(A_1)$.
(2) Every regular function algebra is weakly regular.
(3) The ‘Tomato-can algebra’ is weakly regular [5, 12], but not approximately regular and hence not regular.

(4) Approximate regularity and weak regularity are independent concepts as the tomato-can algebra is weakly regular but not approximately regular and the disk algebra on the unit circle $\Gamma$ is approximately regular on $\Gamma$ but not weakly regular on $\Gamma$.

**Theorem 3.4.** $A_1 \times A_2$ is weakly regular on $\Delta(A_1 \times A_2)$ if and only if $A_1$ and $A_2$ are weakly regular on $\Delta(A_1)$ and $\Delta(A_2)$ respectively.

**Proof.** Suppose $A_1$ and $A_2$ are weakly regular on $\Delta(A_1)$ and $\Delta(A_2)$ respectively. Let $K \subsetneq \Delta(A_1 \times A_2)$ be closed. Suppose $\mathrm{K} \cap \Delta(A_1) \neq \emptyset \neq \mathrm{K} \cap \Delta(A_2)$. Then $\mathrm{K} \cap \Delta(A_1)$ and $\mathrm{K} \cap \Delta(A_2)$ are proper closed subsets of $\Delta(A_1)$ and $\Delta(A_2)$ respectively. Since $A_1$ and $A_2$ are weakly regular, there exist $f_1 \in A_1$ and $f_2 \in A_2$ such that $f_1 \not\equiv 0$, $f_2 \not\equiv 0$, $\widehat{f_1}_{|\mathrm{K} \cap \Delta(A_1)} \equiv 0$ and $\widehat{f_2}_{|\mathrm{K} \cap \Delta(A_2)} \equiv 0$. Consider $h = (f_1, f_2)$. Then $h \in A_1 \times A_2$, $h \not\equiv 0$ and $\hat{h}_{|\mathrm{K}} \equiv 0$.
If $K \subsetneq \Delta(A_1)$ (or $K \subsetneq \Delta(A_2)$) is closed, then $h = (f_1, 0) \in A_1 \times A_2$ ($h = (0, f_2) \in A_1 \times A_2$) suffices the purpose.
If $K = \Delta(A_1)$, then $h = (0,1) \in A_1 \times A_2$ and if $K = \Delta(A_2)$,, then $h = (1,0) \in A_1 \times A_2$ suffices the purpose. Hence $A_1 \times A_2$ is weakly regular on $\Delta(A_1 \times A_2)$.

Conversely, suppose that $A_1 \times A_2$ is weakly regular on $\Delta(A_1 \times A_2)$. We shall show that $A_1$ is weakly regular on $\Delta(A_1)$. Let $K \subsetneq \Delta(A_1)$ be a closed set. Then it is clear that $K \cup \Delta(A_2) \subsetneq \Delta(A_1 \times A_2)$ is also a closed set. Since $A_1 \times A_2$ is weakly regular, $\exists\, h = (f_1, f_2) \in A_1 \times A_2$ such that $\hat{h}_{|\mathrm{K} \cup \Delta(A_2)} \equiv 0$. Then $\widehat{f_1}_{|\mathrm{K}} \equiv 0$ and $\widehat{f_2}_{|\Delta(A_2)} \equiv 0$. Since $\widehat{f_2} = 0$ on $\Delta(A_2)$, $f_2 \equiv 0$. Since $h \not\equiv 0$ we must have $f_1 \not\equiv 0$. So $A_1$ is weakly regular on $\Delta(A_1)$. Similarly, $A_2$ is weakly regular on $\Delta(A_2)$. ■

The above theorem can also be established by employing the definition introduced by Feinstein and Somerset together with the following theorem.

**Theorem 3.5.** [11] Let $\tau_{hk}^{A_1}$ and $\tau_{hk}^{A_2}$ be hull-kernel topologies on $\Delta(A_1)$ and $\Delta(A_2)$ respectively. Let $\tau_{hk}^{A_1 \times A_2}$ be the hull-kernel topology on $\Delta(A_1 \times A_2)$ and $\tau$ be the sum topology on $\Delta(A_1 \times A_2)$ induced by $\tau_{hk}^{A_1}$ and $\tau_{hk}^{A_2}$. Then $\tau_{hk}^{A_1 \times A_2} = \tau$.
**Proof.** In order to prove the theorem, one should prove the following.
(1) $k(E \cup F) = k(E) \times k(F)$, where $E \subset \Delta(A_1)$, $F \subset \Delta(A_2)$.
(2) $h(I \times J) = h(I) \cup h(J)$, where $I$ and $J$ are proper ideals of $A_1$ and $A_2$ respectively.

## Conclusion

The Cartesian product of function algebras retains approximate regularity, approximate normality, and weak regularity whenever both component algebras possess the respective property.

## Conflict of Interest

The author declares that there is no conflict of interest.

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