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Definition of Perfect Shape and Their Values in n-Dimensional World

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Abstract

This paper introduces the concept of the perfect shape, defined as an n-dimensional figure whose enclosed measure and boundary measure are connected through a differential relationship. Specifically, a shape is perfect when the derivative of its n-dimensional quantity with respect to its defining size parameter equals its (n−1)-dimensional quantity. Through geometric and differential analysis, the paper demonstrates that this condition holds exclusively for hyperspheres—figures whose boundary points are equidistant from a central origin. Circles and spheres thus serve as lower-dimensional instances of this universal form. Building upon this foundation, the paper proposes a conjecture linking the measures of perfect shapes across consecutive dimensions. Observing that differentiation and integration connect adjacent dimensions, it is hypothesized that higher-dimensional expressions of the hypersphere can also be derived by multiplying the measure of the n-dimensional figure by 2^n, reflecting a recursive geometric pattern among perfect shapes. This framework offers a new perspective on dimensional growth and suggests that higher-dimensional geometry may be constructed through the calculus of adjacent dimensions. Although this study presents a theoretical conjecture, future research will aim to validate the proposed 2^n-scaling relationship through topological data analysis and AI-based modeling, providing a computational approach to bridging analytic and geometric views of higher-dimensional space.

Keywords: Perfect shape; n-dimensional geometry; Dimensional recursion; Hypersphere.

1. Introduction

Geometric quantities (e.g., length, area, volume) are measurements of shapes in space. Understanding how these quantities are related reveals fundamental relationships among dimensions of space. For certain shapes in n-dimensional space, these quantities are linked through differentiation: the rate of change of its n-dimensional quantity with respect to its defining size parameter is equal to its $(n-1)$-dimensional quantity. For instance, when a circle (a two-dimensional shape) expands, the rate at which its area $A = \pi r^2$ (2D measure) increases with respect to its radius r is equal to its circumference $C = 2\pi r$ (1D measure). Likewise, the derivative of a sphere's volume $V = \frac{4}{3}\pi r^3$ yields its surface area $S = 4\pi r^2$.

However, this derivative relationship does not extend to all shapes. The derivative of a square's area is not equal to its perimeter, just to name one of such shapes that fail to maintain this relationship.

This observation motivates the definition of a new category of shapes which we call perfect shapes. A shape is perfect when the rate of change of its enclosed measure is exactly equal to its boundary measure. This condition can only be satisfied by shapes whose every boundary point is equidistant from a central origin—namely circles, spheres, and their higher-dimensional analogues, the n-dimensional hyperspheres—whereas shapes lacking uniform radial symmetry do not satisfy.

Building on this concept, we explore a potential pattern linking the measures of perfect shapes across consecutive dimensions. Observing that the surface area of a sphere ($S = 4\pi r^2$) is four times the area of a circle ($A = \pi r^2$), we hypothesize that this relationship generalizes as follows: multiplying an n-dimensional measure by 2^n and integrating with respect to the size parameter may yield the corresponding $(n+1)$-dimensional measure. This proposed 2^n pattern suggests a recursive structure among perfect shapes, providing an intuitive pathway toward expressing the volumes of hyperspheres in arbitrary dimensions.

To visualize this idea, we introduce the tennis ball analogy. A square of side length $4r$ can contain four circles of radius r. When conceptually "folded" into 3D—analogous to how a tennis ball is constructed—the 4 circles form a sphere. This arrangement suggests that the surface of a sphere is formed by $2^2 = 4$ circles. Extending this reasoning, a cube of side length $4r$ can contain $2^3 = 8$ spheres, hinting at a structural pattern across dimensions. While this analogy is not a formal proof, it offers a tangible visualization for the hypothesized dimensional progression of perfect shapes.

Finally, we note that a well-established analytical formula, derived from the Gamma function, already exists for the volume of n-dimensional hyperspheres. Our proposed method does not seek to contradict that formulation but to offer an alternative geometric perspective based on differential relationships and dimensional growth patterns. The theoretical framework presented here constitutes the first part of this research. Future work will focus on computational validation and numerical comparison between the established Gamma-based model and our hypothesized 2^n-scaling approach.

2. Perfect Shape

2.1. Definition

An n-dimensional shape is said to be *perfect* if the rate of change of its n-dimensional quantity with respect to its defining size parameter equals its $(n-1)$-dimensional quantity. This can be expressed as:

$$\frac{d(n\text{−}dimensional\ measure)}{d(size\ parameter)} = (n-1)\text{-dimensional measure}$$

For an n-dimensional shape, its n-dimensional quantity represents the measure of the space enclosed by the shape; for example, a 2D shape encloses a surface, and its 2D quantity corresponds to the area of that surface.

The defining size parameter is a 1D quantity that characterizes the scale of the shape and is used to calculate the shape's geometric quantities; for example, a radius is the defining size parameter for a circle or sphere, while an edge is one for a square or cube.

The $(n-1)$-dimensional quantity represents the measure of its boundary; for example, a perimeter is the 1D boundary measure for 2D shapes, and a surface area is the 2D boundary measure for 3D shapes.

In other words, a shape is perfect if the instantaneous rate of change of its enclosed measure is exactly equal to its boundary measure.

Example: A circle is a perfect shape.

In 2D, a circle belongs to this classification. A circle's enclosed measure is the area $A = \pi r^2$. Deriving the area with respect to the radius r gives the circle's boundary measure, circumference $C = 2\pi r$. The differentiation $\frac{d}{dr}\pi r^2 = 2\pi r$ proves that $\frac{dA}{dr} = C$, proving that a circle is a perfect shape.

2.2. Conditions for Perfection

Consider an n-dimensional shape. Let R be its defining size parameter, S_{n-1} be its $(n-1)$-dimensional surface "area" or boundary, and $V_n(R)$ be its n-dimensional volume enclosed by the shape.

Proposition 2.2.1. If an n-dimensional shape satisfies the perfect-shape condition

$$\frac{dV_n(R)}{dR} = S_{n-1},$$

then the distance from the chosen center to the boundary must be the same in every direction.

Proof by example:

1. Circle (example of a perfect shape).

Area of a circle: $A = \pi r^2$. Differentiate with respect to radius r:

$$\frac{dA}{dr} = 2\pi r.$$

Write in the differential form, $dA = 2\pi r \cdot dr$. The coefficient $2\pi r$ is exactly the circle's circumference. Thus a small *uniform radial* increase dr adds an area equal to $perimeter \times dr$. The circle expands uniformly because every boundary point is at the same distance r from the center.

2. Square (counterexample).

Let x be the orthogonal distance from a square's center to the midpoint of its edge. Writing area in terms of x gives $A = 4x^2$, so

$$dA = 8xdx.$$

Let y be the distance from the square's center to corner ($y = \sqrt{2x}$). Writing area in terms of y gives $A = 2y^2$, so

$$dA = 4ydy = 4(\sqrt{2x})dy.$$

The two relations give different increments dx and dy to produce the same area change dA. Suppose $x = 1$ and $dA = 1$. Then

$dx = \frac{1}{8}$ but $dy = \frac{1}{4\sqrt{2}}$,

so the distance toward an edge and the distance toward a corner must change at different rates to produce the same change in area. Hence there is no single uniform radial increment that produces an area increment equal to $perimeter \times increment$.

3. Conclusion.

The circle example exhibits a single radial parameter r for which $dA = (perimeter)dr$. The square example shows that when boundary distances differ by direction the same identity cannot hold with a single size parameter. Therefore, for the differential equality to hold uniformly, the distance from the chosen center to the boundary must be constant in every direction.

If every boundary point is at distance R from the center, a small outward increment dR produces a thin shell whose first-order n-volume is $(boundary\ measure) \times dR$. Thus, for such radially constant shapes (the n-balls and hyperspheres) the identity

$$\frac{dV_n(R)}{dR} = S_{n-1}$$

holds: the derivative of the enclosed n-volume with respect to R equals the $(n-1)$-measure of the boundary.

2.3. Perfect Shapes in 3D

Proposition 2.3.1. With the reason for the derivative - integral relationship proven between the area and circumference of the circle through the methods stated in **Proposition 2.2.1**, the same exact relationship remains consistent moving onto the formulaic expression of the sphere in the third dimension (its volume) to the expression of the sphere in the second dimension (its surface area).

With the volume equation of the sphere,

$$V = \frac{4}{3}\pi r^3$$

Differentiating the given equation would give,

$$dV = 4\pi r^2$$

And given SA represents the surface area of a sphere,

$$4\pi r^2 = SA$$

It would lead to this equation after plugging in the respective values,

$$dv = SA$$

Proposition 2.4.1. Through this relationship shown in **Proposition 2.3.1**, the sphere becomes the only geometric shape that can yield a sound result of overall change in volume found through the net change theorem.

Example: Given that you are trying to find the net change between the volume of a sphere when radius equals to 1 compared to when the radius is set to 2, the volume of the sphere with radius of 1 would be expressed as,

$$V(1) = \frac{4}{3}\pi(1)^3$$

$$V(1) = \frac{4}{3}\pi$$

And the volume of the sphere with radius of 2 would be expressed as,

$$V(2) = \frac{4}{3}\pi(2)^3$$

$V(2) = \frac{32}{3}\pi.$

Thereby, the net change of the volume of the given spheres would be

$$V(2) - V(1)$$

$\frac{32}{3}\pi - \frac{4}{3}\pi = \frac{28}{3}\pi.$

Doing this process again but through the Net change theorem, given the same conditions,

$$\int_1^2 S(r)\,dr = V(2) - V(1)$$

Because of the derivative-integral relationship between the volume and surface area of the sphere, this equation fits in soundly with the formulaic expression of the shape,

$$\int_1^2 \quad 4\pi r^2 dr = \frac{4}{3}\pi(2)^3 - \frac{4}{3}\pi(1)^3$$

$\int_1^2 \quad 4\pi r^2 dr = \frac{32}{3}\pi - \frac{4}{3}\pi = \frac{28}{3}\pi.$

Through the work shown, with the equation for the surface area and volume of the sphere, the net change theorem can be utilized to find the net change of the volume as the value resulting from the first and second calculations, despite different methods being used, are equal.

Lemma 2.5.1. The existence of the relationship shown in **Proposition 2.4.1.** is proven by the fact that, because every single line connecting the CenterPoint of a sphere to the edge of the sphere is uniform in length, if the volume of the sphere increases or decreases at a given constant rate, each vector within said sphere will uniformly increase or decrease at a shared rate proportional to the rate of change of the volume:

With the volume equation of the sphere,

$$V = \frac{4}{3}\pi r^3$$

Differentiating the given equation in respect to time would give the related-rates equation of,

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

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And since, given SA represents the surface area of a sphere,

$$4\pi r^2 = SA$$

It would lead in the equation

$$\frac{dV}{dt} = SA\,\frac{dr}{dt}$$

Thereby, because the rate of change of the volume ($\frac{dv}{dt}$) is the equivalent to the rate of change of the radius ($\frac{dr}{dt}$) multiplied by $4\pi r^2$ (SA) if the volume of the sphere expands at a certain rate, it would expand $4\pi r^2$ (SA) times faster than its radius. And because the radius is uniform at every point within the sphere, this relationship between the rate of the change of the volume and any vector going from the center of the shape to the very edge remains constant regardless of which point is being used, given that it is within the same sphere.

Example: Given that $\frac{dV}{dt} = 1$ and $r = 1$, it creates the equation

$1 = 4\pi(1)^2\frac{dr}{dt}$,

leading to

$\frac{dr}{dt} = \frac{1}{4\pi}$.

Because of the fundamental nature of a sphere with it having a uniform length from every point from its center of shape to the outer edge, all vectors of the given sphere under given circumstances will be moving at a rate of $\frac{1}{4\pi}$ in relation to time without exception.

However, with any other geometric shapes' expressions as a 3D value, if its volume is increasing at a constant rate, it becomes an impossibility for every vector within the shape to be going at a uniform rate.

Counterexample: Using the formula of the cube,

$$V = x^3$$

If the equation was to be changed to give the volume of the cube in relation to a line from its center point of the overall shape to the center point to a side of the cube given as s,

$$V = (2s)^3$$

Differentiating the given equation in respect to time would give the related-rates equation of,

$$\frac{dV}{dt} = 24s^2\frac{ds}{dt}$$

If the rate of change of volume in respect to time is given as

$$\frac{dV}{dt} = 1,$$

and the distance from the cube's center to the midpoint of one face is given as

$$s = 1,$$

It creates the equation

$$1 = 6(1)^2\frac{ds}{dt}$$

making it so that

$$\frac{ds}{dt} = \frac{1}{6}.$$

Yet, if the equation of the cube was changed to give the volume of the cube in relation to a line from its center point to a corner of its side (c), it would become

$$V = (\frac{\sqrt{2}c}{2})^3.$$

Differentiating the given equation in respect to time would give the related-rates equation of

$$\frac{dV}{dt} = \frac{3c^2\sqrt{2}}{4}\frac{dc}{dt}.$$

If the rate of change of volume in respect to time was given as,

$$\frac{dV}{dt} = 1$$

and the length of the line going from the CenterPoint of the shape to a midpoint on its edge is given as,

$$c = \sqrt{2}$$

making it so the length that c (the line from the center of the cube to a corner of the cube's edges at its midpoint) is the value it would be when s (the line from the center of the cube to the center of a side) is equal to 1 as given in the previous geometric scenario, It creates the equation

$$1 = \frac{3(\sqrt{2})^2\sqrt{2}}{4}\frac{dc}{dt}$$

making it so that the rate of change of c in relation to time is,

$$\frac{dc}{dt} = \frac{4}{6\sqrt{2}}$$

In contrast to the sphere, for the cube, two vectors that are within the exact same shape and having the rate of change of the shape's volume be exactly the same for said shape that the vectors are in, would require different rates of change in order to be consistent in the volume they are in. Therefore, under constant circumstances, an increase in the volume of a cube would result in the ratio between the rate of change of its volume in relation to time to the rate of change of its vectors in relation to time being non-uniform depending on which vectors are being observed.

It is exactly because of this disparity that the derivative-integral relationship only occurs between the volume and surface area of spheres and no other shapes. It is the only geometric shape whose distance from the center of the shape to the outer edge is uniform at every point.

Thereby through the exact same way as it was established in the formulaic expression of the circle in the 2nd dimension with its area and circumference stated in **Proposition 2.1**, the same integral-derivative relation carries on into the 3rd dimension. Thereby, the sphere maintains its definition as a perfect shape beyond the second dimension.

3. Formulaic Connection Between Adjacent Dimensions

Corollary 3.1. Having established the fact that there exists a derivative - integral relationship between the circumference and area of the circle as well as between the surface area and volume of a sphere, in both the 2nd and 3rd dimension (**Proposition 2.2.1, Proposition 2.3.1**), it can be said that there is a consistent relationship between the formulaic expression of the circle and the sphere between its expression in the nth dimension as well as its expression in the n-1 dimension.

For a circle, which exists in the 2nd dimension, the formulaic expression of it in the correlating dimension would be, πr^2 while its expression within the dimension below it would be $2\pi r$.

For a sphere, which exists in the 3rd dimension, the formulaic expression of it in the correlating dimension would be, $\frac{4}{3}\pi r^3$ while its expression within the dimension below it would be $4\pi r^2$.

Observation 3.2. Through these relationships, an overlap gets created in the second dimension between the expression of a circle in its correlating dimension, πr^2, and the expression of the sphere within the dimension below it (the 2nd dimension), $4\pi r^2$.

Both of these expressions can be created in terms of the second dimension but are representing shapes that are in differing dimensions as πr^2 represents a shape in the correlating dimension, the 2nd, and $4\pi r^2$ represents a shape above the dimension it is expressed for, the 3rd.

With this new relationship, a factual observation between these two expressions can be made as in, the surface area of a sphere equates to the area of a circle multiplied by 4, given that the radius remains consistent.

$$4 \times \pi r^2 = 4\pi r^2$$

Conjecture 3.3. This relationship arises from the mathematical connection between the equations describing a circle in dimension n (expressed both in n-D and (n–1)D) and those describing a sphere in dimension (n+1) (expressed both in (n+1)-D and n-D). This connection is formed by multiplying the formula of the circle in dimension n by 2^n to result in the formulaic expression of the sphere at nD.

Formulation 3.4. To quantify this, at the second dimension, meaning $n_1 = 2$, πr^2 would be representative of the circle in its correlating dimension at $n = 2$,

$2\pi r$ would be representative of the circle expressed as a value in the dimension below the given value (2) at $n = 1$ or at $n - 1$.

At the third dimension,

$$n_1 + 1 = n_2$$
$$n_2 = 3$$

Meaning that when n = 3 or when $n = n_2 + 1$, $\frac{4}{3}\pi r^3$ would be representative of the sphere in its correlating dimension at $n = 3$, $4\pi r^2$ would be representative of the sphere expressed as a value in the dimension below the given value (3) at $n = 2$ or at $n_2 - 1$.

As stated previously **(Conjecture 3.3)**, when looking at the expressions for when $n = 2$, the expression of the shape at n_1 within the correlating dimension, is multiplied by 2 to the power of n,
$2^n \times \pi r^2 = 4\pi r^2$ is found.

Since for both πr^2 and $4\pi r^2$, $n = 2$,

$$2^2 \times \pi r^2 = 4\pi r^2$$
$$4 \times \pi r^2 = 4\pi r^2$$

Resulting in the same equation as previously stated **(Observation 3.2)**

Using the same method, it can be translated to make a prediction of the equation of a sphere in the 4th dimension.

Formulation 3.5. As previously established, at the third dimension,

$$n_2 = 3$$

meaning that when n equals 3, the equation relating directly in terms of that dimension would be $\frac{4}{3}\pi r^3$. And to create the same shape but quantified in the dimension below (n-1), the equation $4\pi r^2$ would be created.

Moving forward to the dimension above at,

$$n_3 = 4$$

Or, $n_3 = n_2 + 1$

As stated prior **(Conjecture 3.3)**, within this dimension there would be an expression to describe the shape in a way to directly relate the equation to the dimension it exists in at n_3 but also a way in which to express the shape created in the 4th dimension but quantified at $n_3 - 1$ or at n_2, the 3rd dimension as $n_2 = 3$.

Thereby, using the equation of the sphere within the 3rd dimension quantified directly related in the equivalent n value at $n_2 = 3$, ($\frac{4}{3}\pi r^3$) the method stated can predict the equation of the sphere in the 4th dimension quantified at $n_3 - 1$ or at $n_2 = 3$, giving:

$$2^n \times \frac{4}{3}\pi r^3$$

Since $n_2 = 3$, it would result in,

$$2^3 \times \frac{4}{3}\pi r^3 = \frac{32}{3}\pi r^3$$

4. Application of The Derivative-Integral Relationship in the 4th Dimension

Corollary 4.1. With having found the formulaic expression of the sphere in the 4th dimension or when n=4, quantified at the dimension below (n-1 or at n=3), through the integral derivative relationship which was previously proven to exist at every dimension for a sphere **(Proposition 2.3.1, Proposition 2.2.1)**, applying this method, since at n=4, the expression $\frac{32}{3}\pi r^3$ is the value quantified at n-1 rather than directly related to n, through the integration of the expression, the value quantified at n can be found.

$$\int \frac{32}{3}\pi r^3 \, \mathrm{dr} = \frac{8}{3}\pi r^4$$

Through this it can be said that in the 4th dimension when n=4, the shape of the sphere can be formulaically quantified within the dimension below (n-1) as $\frac{32}{3}\pi r^3$ as well as directly quantified at n=4 by $\frac{8}{3}\pi r^4$.

Observation 4.2. If 2 axes were created that would represent the plane of the 2nd dimension (n=2), as the 2nd is characterized by length and wide (x and y axis), if the area of the circle (πr^2) were to be put into each quadrant, the combination of all would equal the surface area of a sphere ($4\pi r^2$).

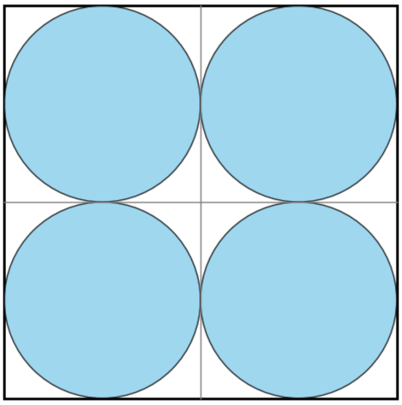

Figure 1. Square containing 2^2 circles.

Observation 4.3. In the same logic, if 3 axes were created that would represent the plane of the 3rd dimension (n=3) as the 3rd dimension is characterized by length, width, and height (x, y, and z), if the volume of the sphere ($\frac{4}{3}\pi r^3$) were to be put into each quadrant, the combination of all would equal the equation that was found using the previous method stated that would represent the formula for the sphere in the 4th dimension but quantified in the dimension below (n-1) as $\frac{32}{3}\pi r^3$

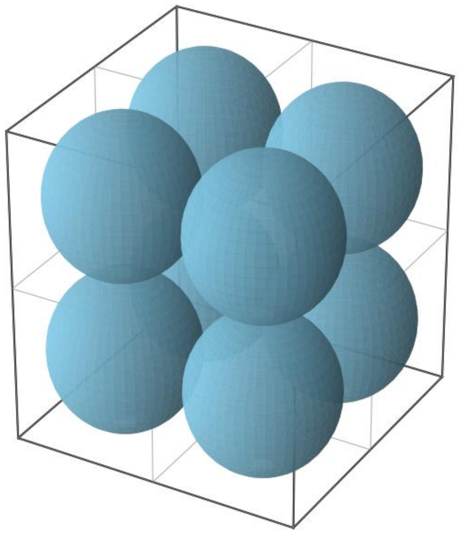

Figure 2. Cube containing 2^3 spheres.

Figure 1 demonstrates that the visual relationship between the two expressions at n=2 and at n-1 are that the value of the expression at the given value of n (πr^2) should fill in every quadrant that is created by the axes of the dimension at the given n (in this case n=2) in order to be equal to the value of the expression of the shape in the next dimension quantified at n-1 at n=3 ($4\pi r^2$).

Translating this to the next value of n at 3, as shown in figure 2, the new axes of z doubles the number of quadrants making it so that if each quadrant is filled with the shape of the expression quantified at n at n=3 ($\frac{4}{3}\pi r^2$) or the volume of the sphere, the combined value would equate to $8 \times \frac{4}{3}\pi r^2 = \frac{32}{3}\pi r^2$ or $2^n \times \frac{4}{3}\pi r^2 = \frac{32}{3}\pi r^2$. Therefore, since each new axis created within the plane would double the number of quadrants within the given plane, the visual representation of this phenomenon supports the formulaic expression that was found **(Formulation 3.5)**

As previously established **(Observation 3.2)**, the expression of $\pi r^2 \times 4 = 4\pi r^2$ is one that can be used as a factual explanation of the connection between the expressions of the surface area of the sphere and the area of the circle. Furthermore, (First figure) is a way to accurately represent this relationship visually in a method that fits within the scope of the explanation, maintaining the fact that it is simply a factual observation. Using the exact same method and logic as used in the first figure, (second figure) is found. And if the image was to be translated to a formula, the visuals in (figure second) matches the established equation of $\frac{4}{3}\pi r^3 \times 8 = \frac{32}{3}\pi r^3$.

Conjecture 4.4. Thereby, since there will be expressions for a shape in both dimensions but quantified in terms of a single dimension as stated in **(Observation 3.2)**, an overlap gets created between every dimension in which in one dimension, there will exist two expressions, one that directly represents the shape at the dimension the expression is quantified in as well as one which represents the shape in the dimension above, a possible way to visually represent the connection between the expressions for the same shape in adjacent dimensions would be that: Through the combination of the values of the shape that is resulting from the expression of the shape in the lower dimension quantified in direct relation to that dimension multiplied by the number of quadrants which is created by the axes that arises within the lower dimension of two given dimensions, the value of the shape in the higher of the two dimensions expressed at n-1 will be found.

Through this, it creates a consistent explanation both formulaically and visually on how the expression that represents a shape at dimension n_1 quantified at n is connected to the expression that represents the shape at dimension n_2 quantified at n-1. As stated previously **(Conjecture 3.3),** the formulaic explanation results in the multiplication of the shape at dimension n_1 quantified at n multiplied by 2^n in order to find the expression of n_2 at n-1. This relates directly to the visual explanation as each new axis that is added to the plain would result in a doubling of the amount of quadrants which exits, meaning that with every new dimension, the quadrants will double, meaning the number of shapes multiplied to find the n-1 value of the shape in the next dimension will double, or in other words, multiplied by 2^n.

5. Conclusion

This paper defined the perfect shape as the sphere due to its derivative-integral relationship across every value of n. Furthermore, through this established relationship, the paper proposed the conjecture that higher-dimensional formulas for the sphere can be derived through this relationship as well as through the multiplication of the expression of the shape at dimension n expressed in direct relation to that dimension with 2^n to find the expression of the shape in the next dimension expressed at n-1 due to the overlap in n-values. This proposal supports the idea that higher dimensional geometry can be derived through the calculus of adjacent dimensions (can site the paper you gave maybe...?) Though unlike previous works, this paper gave an abstract conjecture on the inference of what the formulaic expression of the sphere in the 4th dimension would be. Though currently an abstract conjecture, the next step for research that will proceed this paper is to topologically prove this idea through topological data analysis run through AI modeling.