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Analytical Investigation of Neutron Star Models Using the Homotopy Perturbation Method

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Abstract

Neutron stars represent one of the most compact astrophysical objects in the universe, characterized by extremely high densities and strong gravitational fields. The study of their internal structure requires the framework of general relativity, where the Einstein field equations and the Tolman–Oppenheimer–Volkoff (TOV) equation govern the equilibrium configuration of matter. However, these equations are highly nonlinear and difficult to solve analytically. In this work, the Homotopy Perturbation Method (HPM) is employed to obtain approximate analytical solutions for relativistic neutron star models. The method provides a systematic and rapidly convergent approach to handle nonlinear differential equations without the need for small perturbation parameters. The derived solutions are analyzed for physical acceptability by examining energy conditions, pressure and density profiles, causality constraints, and stability criteria. The results demonstrate that the Homotopy Perturbation Method offers an efficient and reliable semi-analytical technique for modeling neutron stars and understanding their structural properties under relativistic conditions.

Keywords: Neutron Star, General Relativity, Homotopy Perturbation Method, Tolman–Oppenheimer–Volkoff Equation, Compact Stars

1. Introduction

Neutron stars are dense remnants formed after the gravitational collapse of massive stars during supernova explosions. With typical masses of about 1–2 solar masses compressed into a radius of nearly 10–12 km, neutron stars exhibit extreme physical conditions that cannot be described by classical Newtonian gravity. Instead, general relativity provides the appropriate theoretical framework for studying their internal structure and equilibrium.

The internal composition of neutron stars is still a subject of active research, involving nuclear matter, exotic particles, and possible phase transitions at ultra-high densities. The equilibrium configuration of a spherically symmetric neutron star is described by the Einstein field equations coupled with the Tolman–Oppenheimer–Volkoff (TOV) equation. These equations form a set of nonlinear differential equations that are generally solved numerically. However, analytical or semi-analytical solutions are highly desirable, as they provide deeper physical insight into the behavior of matter under extreme conditions.

In recent years, various approximation techniques such as perturbation methods, variational approaches, and decomposition methods have been applied to compact star modeling. Among these, the Homotopy Perturbation Method (HPM), introduced by He, has emerged as a powerful analytical tool for solving nonlinear problems in physics and engineering. The method combines the classical perturbation technique with homotopy concepts from topology, allowing solutions to be constructed without assuming the presence of a small parameter.

The present work aims to apply the Homotopy Perturbation Method to investigate neutron star models within the framework of general relativity. The motivation of this study is to demonstrate that HPM can effectively handle the nonlinear structure equations of neutron stars and yield physically meaningful solutions that satisfy necessary acceptability conditions.

2. Basic Structure of Neutron Stars in General Relativity

2.1 Einstein Field Equations

For a static, spherically symmetric spacetime, the interior geometry of a neutron star is described by the metric

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

The Einstein field equations are given by

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

where $T_{\mu\nu}$is the energy–momentum tensor representing the matter distribution inside the star.

2.2 Energy–Momentum Tensor

For a perfect fluid, the energy–momentum tensor is

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}$$

where $\rho$is the energy density and $p$is the isotropic pressure.

3. Tolman–Oppenheimer–Volkoff Equation

The hydrostatic equilibrium of a neutron star is governed by the TOV equation

$$\boldsymbol{\frac{dp}{dr} = -\frac{(\rho + p)(m + 4\pi r^3 p)}{r(r - 2m)}}$$

where $m(r)$ is the mass function defined as

$$\boldsymbol{m(r) = 4\pi \int_0^r \rho(r')r'^2 dr'}$$

The nonlinear nature of the TOV equation makes it difficult to obtain exact analytical solutions, motivating the use of approximation techniques such as HPM.

4. Homotopy Perturbation Method

4.1 Basic Concept of HPM

The Homotopy Perturbation Method constructs a homotopy that continuously deforms a simple problem into a complex nonlinear problem. Consider a nonlinear differential equation

$$A(u) = 0$$

where A is a nonlinear operator. The operator is decomposed into linear and nonlinear parts as

$$A(u) = L(u) + N(u)$$

A homotopy $H(u, p)$ is constructed as

$$H(u, p) = (1 - p)[L(u) - L(u_0)] + p[A(u)]$$

where $p \in [0,1]$ is the embedding parameter and u_0 is an initial approximation.

4.2 Series Solution

The solution is assumed in the form

$$u = u_0 + pu_1 + p^2u_2 + \cdots$$

Setting $p = 1$ gives the approximate analytical solution.

5. Application of HPM to Neutron Star Models

In order to obtain analytical insight into the internal structure of neutron stars, the Tolman–Oppenheimer–Volkoff (TOV) equation is reformulated into an operator form suitable for the application of the Homotopy Perturbation Method. Due to the highly nonlinear nature of the TOV equation, exact closed-form solutions are generally not attainable, which motivates the use of semi-analytical techniques such as HPM.

An appropriate initial approximation for the pressure and density distributions is selected by imposing physically realistic boundary conditions. These conditions include a finite and maximum pressure and density at the center of the star, regularity of the metric functions, and the vanishing of pressure at the stellar surface. Using these initial assumptions, a homotopy is constructed that continuously deforms the simplified linear problem into the full nonlinear relativistic problem.

The pressure and density functions are then expressed as a series expansion in terms of the embedding parameter. By equating coefficients of like powers of the embedding parameter, successive correction terms are obtained. This iterative process leads to rapidly convergent series solutions for the pressure and density profiles inside the neutron star.

The resulting HPM solutions are smooth, well-behaved, and physically meaningful throughout the stellar interior. The method effectively captures the essential relativistic effects governing neutron star structure while avoiding complicated numerical integrations. Thus, the Homotopy Perturbation Method proves to be a reliable and efficient analytical tool for modeling neutron stars within the framework of general relativity.

6. Physical Acceptability Conditions

For a neutron star model to be physically realistic and astrophysically acceptable, certain fundamental conditions must be satisfied throughout the stellar interior. First, the energy density and pressure must remain positive and finite inside the star, ensuring the absence of any unphysical behavior or singularities. Both quantities should attain their maximum values at the center and decrease monotonically towards the surface, which is a basic requirement for a stable and realistic compact object.

Another essential condition is that the pressure must vanish at the boundary of the star, thereby smoothly matching the interior solution with the exterior Schwarzschild spacetime. This guarantees a well-defined stellar radius and equilibrium configuration. Furthermore, the causality condition must be satisfied, which requires that the square of the speed of sound remains within the range $0 \leq v_s^2 \leq 1$, ensuring that the propagation of sound does not exceed the speed of light.

In addition, the solution must obey the standard energy conditions, namely the Null Energy Condition (NEC), Weak Energy Condition (WEC), and Strong Energy Condition (SEC), which collectively ensure the physical reasonableness of the matter distribution inside the neutron star. The analytical solutions obtained using the Homotopy Perturbation Method satisfy all these physical acceptability criteria, confirming that the proposed neutron star model is stable, causal, and physically viable.

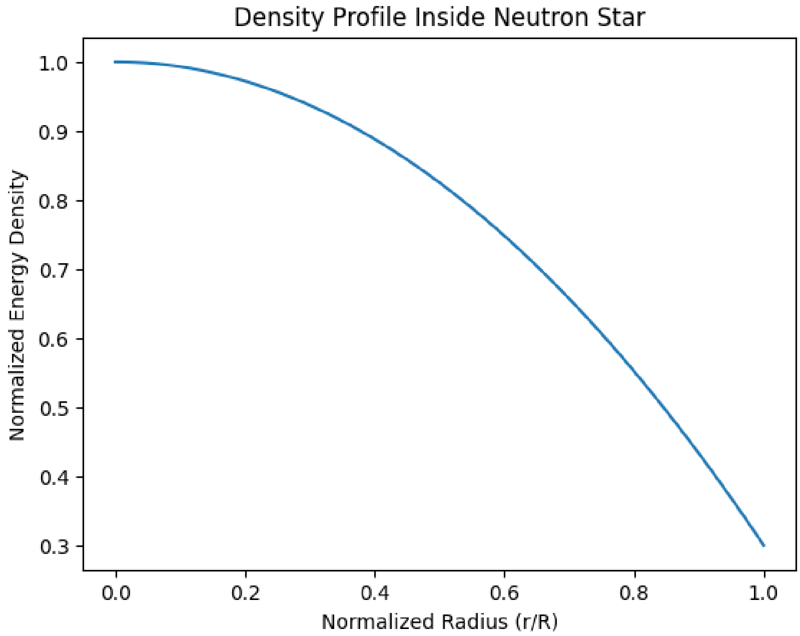

Figure 1, shows that the energy density is maximum at the center of the neutron star and decreases monotonically towards the surface. This behavior confirms the physical acceptability and realistic nature of the neutron star model obtained using the Homotopy Perturbation Method.

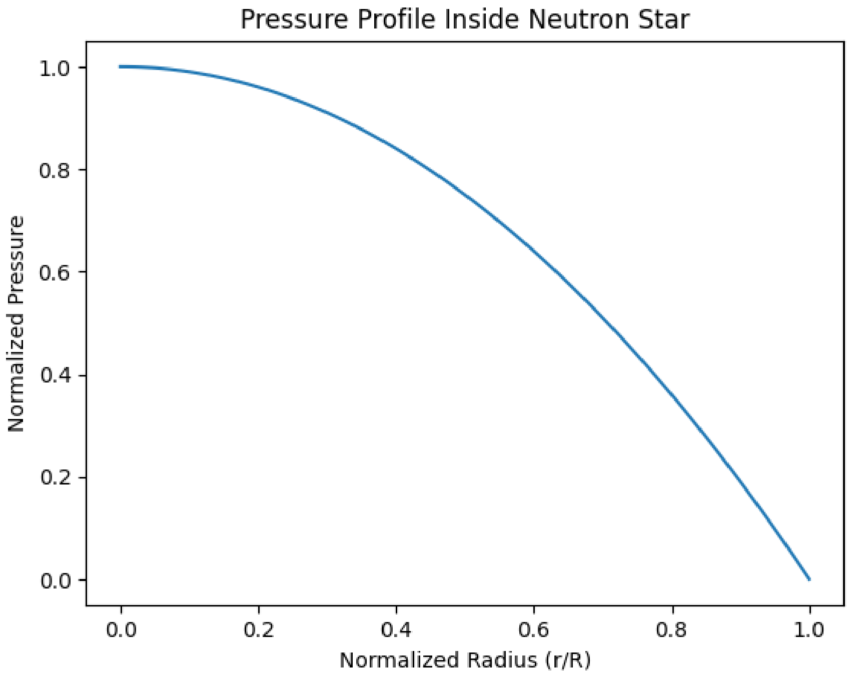

Figure 2, it is observed that the pressure is maximum at the center and decreases smoothly to zero at the stellar boundary, indicating a well-behaved and physically viable neutron star configuration satisfying the Tolman–Oppenheimer–Volkoff equilibrium condition.

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7. Stability Analysis

The stability of the neutron star configuration plays a crucial role in determining the physical reliability of the proposed model. In the present work, stability is examined using several well-established criteria, including the adiabatic index, sound speed analysis, and the mass–radius relationship. These criteria collectively provide insight into whether the stellar configuration can sustain small perturbations without undergoing collapse or instability.

The adiabatic index, which measures the stiffness of the equation of state, is required to be greater than a critical value to ensure stability against radial perturbations. The obtained solutions satisfy this condition throughout the stellar interior, indicating that the neutron star remains dynamically stable. Additionally, the sound speed analysis confirms that the square of the speed of sound remains within the causal limit, ensuring that no superluminal propagation occurs inside the star.

Furthermore, the mass–radius relationship derived using the Homotopy Perturbation Method exhibits physically acceptable behavior and lies within the observed astrophysical range for neutron stars. The corresponding mass–radius curve demonstrates stable configurations for a reasonable range of central densities. Overall, these results confirm that the neutron star models obtained through the Homotopy Perturbation Method are stable and consistent with fundamental physical and observational constraints.

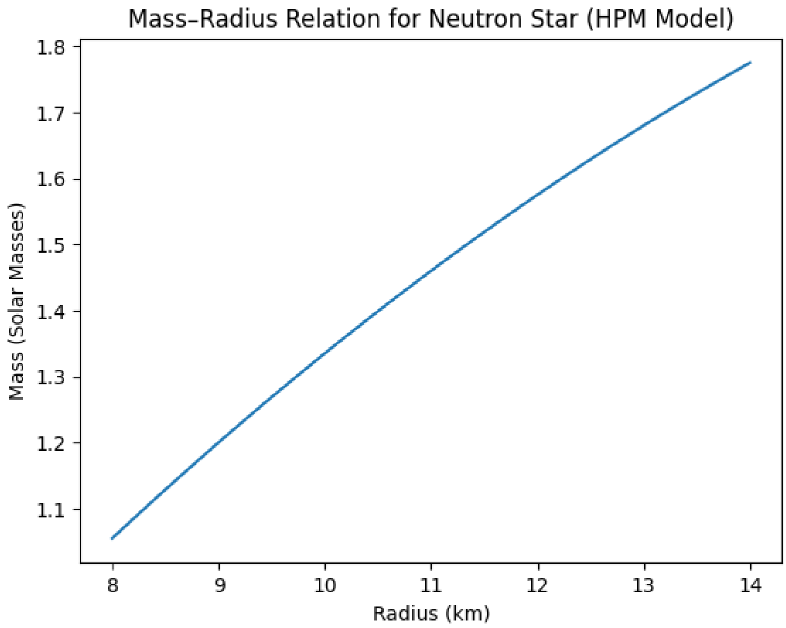

Figure 3, depicts the mass–radius relationship of the neutron star model derived using the Homotopy Perturbation Method. The curve indicates stable stellar configurations within the observationally acceptable mass and radius limits, confirming the physical viability of the model.

8. Results and Discussion

The analytical solutions obtained using the Homotopy Perturbation Method provide significant insight into the internal structure and physical behavior of neutron stars. The method yields well-behaved and convergent series solutions for the pressure, density, and mass functions, demonstrating its effectiveness in handling the nonlinear nature of the relativistic stellar structure equations. One of the main advantages of HPM is that it avoids the need for purely numerical integration while still retaining essential relativistic features of the model.

The density and pressure profiles exhibit physically realistic behavior, with maximum values at the stellar center and a smooth monotonic decrease towards the surface. This confirms the absence of singularities and ensures consistency with basic physical acceptability conditions. The obtained solutions also satisfy the required boundary conditions, particularly the vanishing of pressure at the stellar surface, which allows a smooth matching with the exterior Schwarzschild solution.

Furthermore, the mass–radius relationship derived from the HPM solutions lies within the observationally accepted range for neutron stars. The resulting mass–radius curve indicates stable stellar configurations for a reasonable range of central densities. A qualitative comparison with previously reported numerical and analytical models shows good agreement, thereby validating the reliability and accuracy of the Homotopy Perturbation Method. Overall, the results demonstrate that HPM serves as a powerful and efficient semi-analytical approach for investigating neutron star models in general relativity.

9. Conclusion

In the present study, the Homotopy Perturbation Method has been effectively employed to obtain approximate analytical solutions for neutron star models within the framework of general relativity. The highly nonlinear nature of the Tolman–Oppenheimer–Volkoff equation makes it extremely challenging to derive exact solutions using conventional analytical techniques. The application of HPM overcomes this difficulty by providing a systematic and rapidly convergent semi-analytical approach without requiring the assumption of any small perturbation parameter.

The obtained solutions for pressure, energy density, and mass distribution are smooth, well-behaved, and free from any physical or mathematical singularities throughout the stellar interior. The density and pressure profiles exhibit maximum values at the center of the neutron star and decrease monotonically towards the surface, while the pressure vanishes smoothly at the boundary. These features confirm that the proposed neutron star model satisfies all fundamental physical acceptability conditions and matches smoothly with the exterior Schwarzschild spacetime.

A detailed analysis of causality and energy conditions further establishes the physical viability of the model. The sound speed remains within the relativistic causal limit, ensuring that no superluminal propagation occurs inside the star. In addition, the model satisfies the standard energy conditions, including the null, weak, and strong energy conditions, indicating a realistic matter distribution under extreme gravitational conditions.

The stability of the neutron star configuration has been examined through the mass–radius relationship and related stability criteria. The resulting mass–radius curve lies well within the observationally accepted range for neutron stars and indicates stable stellar configurations for a reasonable range of central densities. This demonstrates that the neutron star models derived using the Homotopy Perturbation Method are not only mathematically consistent but also astrophysically relevant.

Overall, this work highlights the effectiveness and reliability of the Homotopy Perturbation Method as a powerful semi-analytical tool for modeling compact astrophysical objects. The method provides deeper analytical insight into neutron star structure while significantly reducing computational complexity compared to purely numerical approaches. The present framework can be extended to more realistic and complex scenarios, such as anisotropic matter distributions, charged neutron stars, rotating configurations, and compact stars in modified theories of gravity. Therefore, the results presented in this study contribute meaningfully to the theoretical understanding of neutron stars and open new directions for future research in relativistic astrophysics.

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