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Applications of Edge Coloring Graph with Chromatic Index

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Abstract

A proper edge coloring of a graph is an edge coloring such that no two adjacent edges are assigned the same color. The minimum number of colors needed to color edges of G is called the chromatic index $\chi'(G)$ of G. We introduce the new term C_e - V table for finding the inequality between $\chi'(G)$ and the sum of the terms domination number and the chromatic number of the induced sub graph. Here we introduced the applications of the edge coloring with the chromatic number in different dominations.

Keywords: Interval graph, Dominating set, Domination number of the graph G, Edge coloring, Chromatic index, Chromatic index of the vertex induced sub graph, Doubly connected dominating set, k-Regular graph.

1. Introduction

A graph $G=(V,E)$ is an interval graph if the vertex set V can be put into one-to-one correspondence with a set of intervals I on the real line R such that two vertices of G are joined by an edge in E if and only if their corresponding intervals in I have non-empty intersection. That is if $i = [a_i , b_i]$ and $j = [a_j , b_j]$, then i and j intersect means either $a_j < b_i$ *or* $a_i < b_j$. The set I is called an interval representation of G and G is referred to as the intersection graph of I. Also we say that the intervals contain both its end points and that no two intervals share a common end point. The intervals and vertices of an interval graph are one and the same thing. The graph G is connected and the list of sorted end points is given and the intervals in I are indexed by increasing right end points that is $b_1 < b_2 < b_3 < .... < b_n$.

Let $I = \{I_1, I_2, I_3, I_4, ...I_n\}$ be any interval family, where each I_i is an interval on the real line and $I_i = [a_i, b_i]$, for $i = 1, 2, 3, 4, ...n$. Here a_i is called the left end point labeling and b_i is the right end point labeling of I_i . Without loss of generality, we assume that all the end points of the intervals in I are distinct numbers between 1 and 2n. Two intervals i and j are said to be intersect each other if they have non - empty intersection.

2. Preliminaries:

Let $G=(V,E)$ be a graph. A set $D \subseteq V(G)$ is a dominating set of G if every vertex in V/D is adjacent to some vertex in D. A dominating set D of the graph $G(V,E)$ is a non - split dominating set if the induced sub graph $< V-D >$ is connected. A subset D$\subseteq$ V(G) is called dominating set if N[D] = V(G). The minimum cardinality of such a set D is called the domination number $\gamma(G)$ of the graph G. A dominating set D is connected if the subgraph induced by D is connected. The minimum cardinality of connected dominating set D is called the connected dominating number $\gamma_c(G)$ of G. A connected dominating set D is said to be doubly-connected dominating set, if the sub-graph induced by the set V-D is connected. The minimum cardinality of such set is called the doubly connected domination number. It is denoted by $\gamma_{cc}(G)$.

In graph theory, edge coloring involves assigning "colors" to the edges of a graph such that no two adjacent edges (edges sharing a vertex) have the same color.

Proper Edge Coloring: A coloring where no two adjacent edges have the same color.

k-edge-coloring: A coloring that uses exactly k different colors.

Chromatic Index (Edge Chromatic Number): The smallest number of colors needed for a proper edge coloring.

Chromatic Index ($\chi'(G)$): This is the smallest number of colors that can be used in a proper edge coloring of a given graph G.

In graph theory, a regular graph is a graph where all vertices have the same degree. This means every vertex in the graph has an equal number of edges connected to it. If each vertex has a degree of 'k', then the graph is called a k-regular graph. The 4 – Regular domination number was denoted with $\gamma_{4R}(G)$.

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Explanation of The C_e – V Table:

We consider the vertices and the corresponding edge colors in vertical and horizontal rows respectively. We denote the edge colors with C_e and we mapped the colors with the vertices. For the every vertex we consider the set of adjacent vertices with respective mapped colors. For the first vertex we take highest numbered adjacent vertex into the dominating set, remaining vertices which are containing the dominated vertex in the set of adjacent vertices need not to give the dominating vertex in the dominating set and also the dominating vertex itself also doesn't gives the dominated vertex. If any vertex does not contain the dominated vertex in the set of adjacent vertices then also we take the highest numbered adjacent vertex from the set and consider it into the dominating set. These C_e – V table gives the dominating set and the domination number of the dominating set gives the equality with the chromatic index of the graph and induced graph.

3. Main Theorems:

3.1. Theorem:

Let I = { $v_1, v_2, v_3, \ldots, v_n$ } be an n interval family and G is an Edge coloring interval graph corresponding to I. If v_k, v_j, v_i are any three last vertices in the interval family with k as the odd vertex and the vertex v_j belongs to the doubly connected dominating set DCDS and the vertex v_j, contains the second vertex of the interval family, then Doubly connected domination occurs in G by the C_e – V table and also gives $\chi^{/}(G) > \gamma(G) + \chi^{/}(V / DCDS)$ where V/DCDS is the vertex induced sub graph and $\chi^{/}(G)$ is the chromatic index of the graph G as well as $\chi^{/}(V / DCDS)$ is the chromatic index of the vertex induced sub graph V/DCDS.

Proof:

Let I = { $v_1, v_2, v_3, \ldots, v_n$ } be an n interval family and G is an Edge coloring interval graph corresponding to I. If v_k, v_j, v_i are any three last vertices in the interval family and the vertex v_j belongs to the doubly connected dominating set DCDS and the vertex v_j, contains the second vertex of the interval family, the last before vertex i.e., v_j intersects the vertex v_2 but not contains the second vertex v_2, then it may not intersect the first vertex of the interval family and also the domination number of the graph will be increases to dissatisfies the inequality. It must contains the vertex v_2. Then as usual the vertex v_j must intersects all the vertices, as it gives the domination number. Since the vertex v_k was the odd vertex, then the dominating set must gives the doubly connected dominating set with even vertex with the inequality $\chi^{/}(G) > \gamma(G) + \chi^{/}(V / DCDS)$.

In this procedure we also find the doubly connected dominating set of an interval graph with edge coloring towards the Ce – V table as given above.

3.2. Illustration:

The Interval graph *G* with edge coloring is as follows,

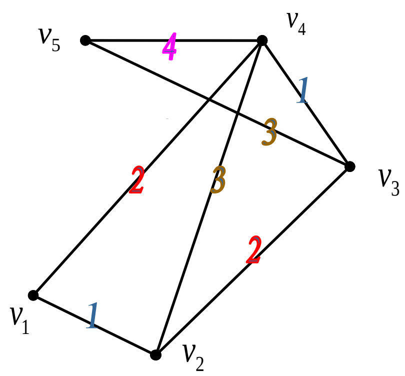

Fig.1: Interval graph *G* with edge coloring

In this interval graph *G* with the edge colors, we can see that the doubly connected dominating set is the even vertex $\{v_4\}$.

The edge colors as follows from the above interval graph *G* is as follows, that no two edges sharing the same color,

$C(v_1,v_2) = 1$, $C(v_1,v_4) = 2$,
$C(v_2,v_3) = 2$, $C(v_2,v_4) = 3$,
$C(v_3,v_4) = 1$, $C(v_3,v_5) = 3$,
$C(v_4,v_5) = 4$

To find the doubly connected dominating set of the above interval graph with edge colors we will introduce the new table namely vertices mapping with the edge colors shortly denoted with $C_e - V$ table as follows,

V/C_e	1	2	3	4
v_1	v_2	v_4	-	-
v_2	v_1	v_3	v_4	-
v_3	v_4	v_2	v_5	-
v_4	v_3	v_1	v_2	v_5
v_5	-	-	v_3	v_4

Now we will find the doubly connected dominating set by using the above $C_e - V$ table as follows,

Step 1:

For the vertex v_1, the set of adjacent vertices with the respective edge colors is $\{v_2, v_4\}$, we consider the vertex v_4 which was the highest numbered adjacent vertex of v_1 into the doubly connected dominating set,

i.e., Doubly Connected Dominating set = $\{v_4\}$

Step 2:

For the vertex v_2, the set of adjacent vertices with the respective edge colors is $\{v_1, v_3, v_4\}$. The highest numbered vertex v_4 exists in the dominating set, so can't consider any set of adjacent vertices for the dominating set.

Step 3:

For the vertex v_3, the set of adjacent vertices with the respective edge colors is $\{v_2, v_4, v_5\}$. The highest numbered vertex v_4 exists in the dominating set, so can't consider any set of adjacent vertices for the dominating set.

Step 4:

The vertex v_4 itself exists in the dominating set, so not consider any vertex in the dominating set.

Step 5:

For the vertex v_5 , the dominating vertex v_4 already exists in the dominating set, so in this step also we can't consider any vertex in the dominating set.

Then the doubly connected dominating set is DCDS = $\{v_4\}$ and the domination number $\gamma_{cc}(G) = 1$ and also the chromatic index of the graph G is $\chi'(G) = 4$

Now the vertex induced sub graph V\DCDS is as follows,

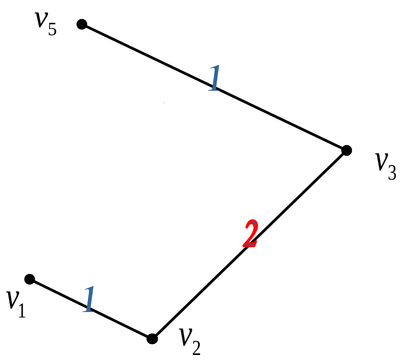

Fig.2: Interval graph induced sub graph V/DCDS with the edge colors

The chromatic index of the induced sub graph V/DCDS is $\chi^{/}(V/D) = 2$

Then clearly we can find the relationship for the doubly connected domination number of the graph G and for the chromatic index of the vertex induced sub graph of G as

$$\chi^{/}(G) > \gamma(G) + \chi^{/}(V/DCDS)$$

Hence the theorem is proved.

3.3. Theorem:

Let I = { $v_1, v_2 ......, v_n$ } be an n interval family and G is an Edge coloring 2 – Regular graph corresponding to I. If v_i, v_j, v_k are the only three vertices in the interval family and the vertex v_k which was the last vertex which was in the 2 – Regular dominating set, then 2 - Regular domination occurs in G by the C_e – V table and also gives the in equality $\chi^{/}(G) > \gamma_{2R}(G) + \chi^{/}(V/4 - RDS)$ where V / 2 - RDS is the vertex induced sub graph and $\chi^{/}(G)$ is the chromatic index of the graph G as well as $\chi^{/}(V/2 - RDS)$ is the chromatic index of the vertex induced sub graph V/2 - RDS.

Proof:

Let I = { $v_1, v_2 ......, v_n$ } be an n interval family and G is an Edge coloring 2 – Regular graph corresponding to I. If v_i, v_j, v_k are the only three vertices in the interval family and the vertex v_k which was the last vertex which was in the 2 – Regular dominating set, then 2 - Regular domination occurs in G by the C_e – V table and also gives the in equality $\chi^{/}(G) > \gamma_{2R}(G) + \chi^{/}(V/4 - RDS)$ where V / 2 - RDS is the vertex induced sub graph and $\chi^{/}(G)$ is the chromatic index of the graph G as well as $\chi^{/}(V/2 - RDS)$ is the chromatic index of the vertex induced sub graph V/2 - RDS. Here we specifically represents that the last vertex must be in the dominating set, of course we can clearly observe that it may the first vertex or the last vertex also satisfies the inequality. By the C_e – V table, the last vertex must be consider into the Regular dominating set otherwise the theorem was not proved according to the table.

In this procedure we also find the 2 - Regular dominating set of 2 - Regular graph with edge coloring towards the Ce – V table as given above.

3.4. Illustration:

The 2 – Regular Graph *G* is as follows,

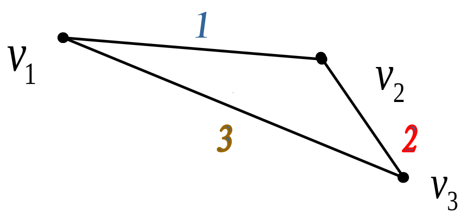

Fig.3: Regular graph *G* with edge coloring

In this Regular graph *G* with the edge colors, we can see that the 2 – Regular dominating set $\{v_3\}$

The edge colors as follows from the above Regular graph *G* is as follows, that no two edges sharing the same color,

$C(v_1,v_2) = 1$, $C(v_1,v_3) = 3$,
$C(v_2,v_3) = 2$

To find the 2 - Regular dominating set of the above Edge color 2 - Regular graph we will introduce the new table namely vertices mapping with the edge colors shortly denoted with C_e – V table as follows,

V/C_e	1	2	3
v_1	v_2	-	v_3
v_2	v_1	v_3	-
v_3	-	v_2	v_1

Now we will find the 2 – Regular dominating set by using the above C_e – V table as follows,

Step 1:

For the vertex v_1, the set of adjacent vertices with the respective edge colors is $\{v_2, v_3,\}$, we consider the vertex v_3 which was the highest numbered adjacent vertex of v_1 into the 2 – Regular dominating set,

i.e., 2 – Regular Dominating set = $\{v_3\}$

Step 2:

For the vertex v_2, the set of adjacent vertices with the respective edge colors is $\{v_1, v_3\}$, the dominating vertex which was the highest numbered vertex already exists in the adjacent vertices, we can't consider any vertex into the 2 – Regular dominating set,

i.e., 2 – Regular Dominating set = $\{v_3\}$

Step 3:

The vertex v_3 already exists in the dominating set, so can't consider any vertex in to the dominating set. Then the 2 – Regular dominating set is 2 - RDS = $\{v_3\}$ and the 2 – Regular domination number $\gamma_{2R}(G) = 1$ and also the chromatic index of the graph G is $\chi^{/}(G) = 3$

Now the vertex induced sub graph V \ 2 - RDS is as follows,

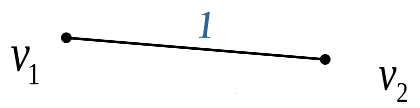

Fig.4: 2 – Regular graph induced sub graph V/2-RDS with the edge colors

The chromatic index of the induced sub graph V/2-RDS is $\chi^{/}(V/4-RDS)=1$

Then clearly we can establish the relationship for the 2 - Regular domination number of the 2 – Regular graph G and for the chromatic index of the vertex induced sub graph of 2– Regular G as

$$\chi^{/}(G) > \gamma_{2R}(G) + \chi^{/}(V/2-RDS)$$

Hence the theorem is proved.

3.5. Theorem:

Let G be an interval graph with edge coloring and with the vertices $\{v_1, v_2, \ldots\ldots, v_n\}$ corresponding to an interval family $I=\{i_1, i_2, \ldots.., i_n\}$. If v_i, v_j, v_k are any three vertices in the interval graph, such that v_i intersects v_k and v_j intersects v_{k+1}, then the chromatic index of the graph G satisfies the inequality $\chi^{/}(G) < \gamma(G) + \chi^{/}(V/D)$ with domination and with the chromatic index of the induced sub graph of the graph G, whenever the dominating set produced by the C_e – V table.

Proof:

Let G be an interval graph with edge coloring and with the vertices $\{v_1, v_2, \ldots\ldots, v_n\}$ corresponding to an interval family $I=\{i_1, i_2, \ldots.., i_n\}$. If v_i, v_j, v_k are any three vertices in the interval graph, such that v_i intersects v_k and v_j intersects v_{k+1}, other than that if the vertex v_i does not intersects the vertex v_k then there was no change in the chromatic number, as it also not effects the chromatic number of the induced sub graph but this connection increases the chromatic number of the graph towards our assumption, but the vertex v_k does not intersects the vertex v_{k+1}, then it leads to the equality of the chromatic number of the graph G and the sum of the domination number and the chromatic number of the induced sub graph of the graph G. So the vertex must intersects the vertex v_{k+1} for the inequality $\chi^{/}(G) < \gamma(G) + \chi^{/}(V/D)$ whenever the dominating set produced by the C_e – V table.

The interval graph G with the edge colors is as follows,

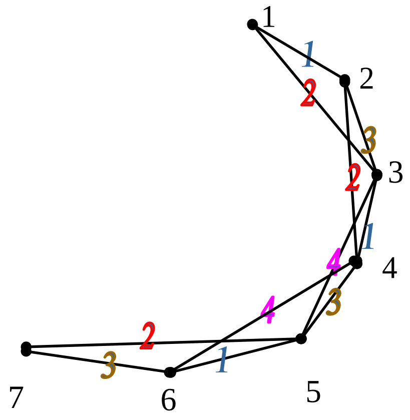

Fig.5: Interval graph *G* with the edge colors

In this Interval graph *G* the dominating set is {3, 7}.

The edge colors as follows from the above interval graph *G*, that no two edges sharing the same color.

$C(v_1,v_2) = 1$, $C(v_1,v_3) = 2$,
$C(v_2,v_3) = 3$, $C(v_2,v_4) = 2$,
$C(v_3,v_4) = 1$, $C(v_3,v_5) = 4$,
$C(v_4,v_5) = 3$, $C(v_4,v_6) = 4$,
$C(v_5,v_6) = 1$, $C(v_5,v_7) = 2$,
$C(v_6,v_7) = 3$

To find the dominating set of the above interval graph with edge colors we will introduce the new table namely vertices mapping with the edge colors shortly denoted with $C_e – V$ table as follows,

V / C_e	1	2	3	4	5
v_1	v_2	v_3	-	-	-
v_2	v_1	v_4	v_3	-	-
v_3	v_4	v_1	v_2	v_5	-
v_4	v_3	v_2	v_5	v_6	-
v_5	v_6	-	v_4	v_3	v_7
v_6	v_5	v_7	-	v_4	-
v_7	-	v_6	-	-	v_5

Now we will find the dominating set by using the above $C_e – V$ table as follows,

Step 1:

For the vertex v_1, the set of adjacent vertices with the respective edge colors is $\{v_2, v_3\}$, we consider the vertex v_3 which was the highest numbered adjacent vertex of v_1 into the dominating set,

i.e., Dominating set = $\{v_3\}$

Step 2:

For the vertex v_2, the set of adjacent vertices with the respective edge colors consists the vertex v_3

Step 3:

The vertex v_3 itself exists in the dominating set

Step 4:

For the vertices v_4 and v_5 the dominating vertex v_3 exists in the set of adjacent vertices with the respective edge colors.

Step 5:

For the vertex v_6 the highest numbered adjacent vertex was v_7 and the dominating vertex v_3 not exists in the set of adjacent vertices with the respective edge colors.

Step 6:

The vertex v_7 itself exists in the dominating set

Then the dominating set is Dominating set = $\{v_3, v_7\}$ and the domination number $\gamma(G) = 2$ and also the chromatic index of the graph G is $\chi^{/}(G) = 4$

Now the vertex induced sub graph V\D is as follows,

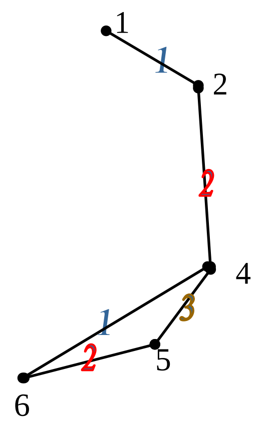

Fig.6: Interval graph *G* with the edge colors

The chromatic index of the induced sub graph V/D is $\chi^{/}(V/D) = 3$

Then clearly we can finalize the relationship for the domination number of the graph G and for the chromatic index of the vertex induced sub graph G as,

$$\chi^{/}(G) < \gamma(G) + \chi^{/}(V/D)$$

Hence the theorem is proved.

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