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Finite Element Analysis of Magneto hydrodynamic Flow of an Incompressible Fluid in a Rectangular Channel

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Abstract

Magnetohydrodynamic (MHD) flow of electrically conducting fluids has significant applications in engineering, metallurgy, cooling systems, and biomedical devices. The present study investigates the steady, fully developed flow of an incompressible viscous fluid in a rectangular channel under the influence of a transverse magnetic field. The governing second-order differential equation is derived from the Navier–Stokes equation by incorporating the Lorentz force term to account for magnetic effects. The resulting boundary value problem is solved numerically using the Finite Element Method (FEM). A weak formulation of the governing equation is developed, and the computational domain is discretized into finite elements. The effect of the magnetic parameter (Hartmann number) on the velocity distribution is analyzed. The results show that increasing the magnetic field strength significantly reduces the fluid velocity due to the resistive Lorentz force. The velocity profile becomes flatter as the magnetic parameter increases. The study demonstrates that the finite element method provides an accurate and efficient numerical tool for analyzing MHD flow problems. The findings may be useful in the design and optimization of industrial systems involving electrically conducting fluids.

Keywords: Magnetohydrodynamic (MHD); Finite Element Method (FEM); Incompressible Fluid; Hartmann Flow; Magnetic Field Effects; Numerical Simulation; Rectangular Channel Flow; Lorentz Force; Computational Fluid Dynamics

1. Introduction

Magnetohydrodynamics (MHD) deals with the study of the motion of electrically conducting fluids in the presence of magnetic fields. Such fluids include liquid metals, plasma, electrolytes, and salt water. The interaction between the magnetic field and the moving conductive fluid generates a resistive force known as the Lorentz force, which significantly alters the flow characteristics (Introduction to Magnetohydrodynamics; Magnetohydrodynamics). Due to its wide range of industrial and engineering applications, MHD flow has attracted considerable attention from researchers over the past several decades.

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One of the classical problems in MHD is the Hartmann flow, which describes the steady flow of a viscous, electrically conducting fluid between two parallel plates under the influence of a transverse magnetic field (Theoretical Hydrodynamics). This model has been widely used to understand magnetic damping effects in channel flows and serves as a benchmark problem for validating numerical methods. The magnetic field suppresses velocity fluctuations and reduces the overall flow rate, leading to a more flattened velocity profile as the Hartmann number increases [1]. MHD flows are important in various engineering applications, including MHD generators, nuclear reactor cooling systems, metallurgical processes, electromagnetic casting, and cooling of electronic devices [2, 3]. In recent years, interest in MHD flow has expanded into biomedical engineering and microfluidic systems, where electromagnetic control of fluid motion offers new possibilities for precision flow regulation [4].

From a mathematical perspective, MHD flow problems are governed by the Navier–Stokes equations coupled with Maxwell's equations. Under simplifying assumptions such as low magnetic Reynolds number and incompressibility, the governing equations reduce to a modified momentum equation containing an additional magnetic force term [5]. Analytical solutions exist only for very simple geometries and boundary conditions. For more realistic configurations, numerical methods are required [6].

Among various numerical techniques, the Finite Element Method (FEM) has proven to be a powerful and flexible tool for solving fluid flow problems, including MHD flows (The Finite Element Method). FEM is particularly advantageous for handling complex geometries and boundary conditions compared to finite difference methods (An Analysis of the Finite Element Method). Its variational formulation allows systematic derivation of weak forms and ensures numerical stability and convergence (Finite Element Procedures). Several researchers have applied FEM to incompressible flow problems and Magnetohydrodynamic systems [7, 8]. The method provides high accuracy even with coarse meshes and can be extended to nonlinear and time-dependent problems. For steady, fully developed channel flows, FEM provides an efficient framework for analyzing velocity distributions and the influence of physical parameters such as viscosity, electrical conductivity, and magnetic field strength [9].

In the context of incompressible viscous flow in a rectangular channel subjected to a transverse magnetic field, the governing equation reduces to a second-order ordinary differential equation with a magnetic parameter commonly expressed in terms of the Hartmann number. The Hartmann number represents the ratio of electromagnetic force to viscous force and plays a central role in characterizing the flow behavior [10]. Despite the availability of analytical solutions for idealized Hartmann flow, numerical investigation remains important for validating computational approaches and extending the model to more general cases. The present study focuses on the finite element formulation of steady MHD flow of an incompressible fluid in a rectangular channel. The weak form of the governing equation is derived, and the domain is discretized using linear finite elements. The effect of the magnetic parameter on velocity distribution is examined in detail. This work aims to provide a simple yet rigorous computational framework suitable for academic research at the postgraduate level while contributing to the broader understanding of magnetically controlled fluid flow systems.

2. Mathematical Formulation

In this study, we consider the steady, fully developed flow of an incompressible, electrically conducting viscous fluid in a rectangular channel subjected to a uniform transverse magnetic field. The flow is assumed to occur between two parallel plates separated by a distance 2h. The x-axis is taken along the direction of flow, and the y-axis is normal to the plates.

2.1 Physical Assumptions

To simplify the mathematical model, the following assumptions are made:

1. The flow is steady and laminar.
2. The fluid is incompressible and Newtonian.
3. The flow is fully developed, so velocity depends only on the transverse coordinate y.
4. A uniform magnetic field B_0 is applied perpendicular to the flow direction.
5. The magnetic Reynolds number is small; therefore, the induced magnetic field is negligible compared to the applied magnetic field.
6. No-slip boundary conditions apply at the channel walls.

Under these assumptions, the velocity field reduces to:

$$V = (u(y), 0, 0)$$

where $u(y)$ is the axial velocity component.

2.2 Governing Equation

The motion of an incompressible conducting fluid in the presence of a magnetic field is governed by the Navier–Stokes equation with an additional Lorentz force term:

$$\rho(V \cdot \nabla)V = -\nabla p + \mu\nabla^2 V + F_m$$

where:

- ρ is the fluid density
- p is pressure
- μ is dynamic viscosity
- F_m is the magnetic (Lorentz) force

For low magnetic Reynolds number, the Lorentz force is given by:

$$F_m = -\sigma {B_0}^2 u$$

where:

- σ is electrical conductivity
- B_0 is magnetic field strength

Since the flow is fully developed and unidirectional, the convective term vanishes. Therefore, the momentum equation in the x-direction reduces to:

$$\mu \frac{d^2u}{dy^2} - \sigma {B_0}^2 u = \frac{dp}{dx}$$

Here, $\frac{dp}{dx}$ is the constant pressure gradient driving the flow.

2.3 Boundary Conditions

The no-slip condition at the channel walls requires:

$$u = 0 \text{ at } y = \pm h$$

2.4 Non-Dimensional Formulation

To generalize the problem, we introduce the following dimensionless variables:

$$Y = \frac{y}{h}, U = \frac{u}{U_0}$$

where U_0 is a characteristic velocity.
Substituting into the governing equation and simplifying, we obtain the dimensionless equation:

$$\frac{d^2u}{dy^2} - Ha^2U = -G$$

where:

$$Ha = B_0 h \sqrt{\frac{\sigma}{\mu}}$$

is the **Hartmann number**, representing the ratio of electromagnetic force to viscous force, and

$$G = \frac{h^2}{\mu U_0}\left(-\frac{dp}{dx}\right)$$

is the dimensionless pressure gradient parameter.

The corresponding dimensionless boundary conditions become:

$$U = 0 \text{ at } Y = \pm 1$$

2.5 Problem Statement

The mathematical problem reduces to solving the following second-order boundary value problem:

$$\frac{d^2u}{dy^2} - Ha^2U = -G$$

subject to:

$$U(-1) = 0, U(1) = 0$$

This boundary value problem will be solved numerically using the Finite Element Method in the next section.

3. Finite Element Method

In this section, the governing boundary value problem derived in the previous section is solved using the Finite Element Method (FEM). The finite element approach is based on transforming the differential equation into its weak (variational) form and discretizing the computational domain into finite elements.

3.1 Weak Formulation

Consider the dimensionless governing equation:

$$\frac{d^2u}{dy^2} - Ha^2U = -G$$

subject to:

$$U(-1) = 0, U(1) = 0$$

To obtain the weak form, we multiply the governing equation by a test function v(Y) and integrate over the domain $\mathbf{\Omega} = [-\mathbf{1}, \mathbf{1}]$:

$$\int_{-1}^{1} v\left(\frac{d^2u}{dy^2} - Ha^2U\right) dY = \int_{-1}^{1} (-G)\, vdY$$

Integrating the first term by parts to reduce the order of differentiation:

$$\int_{-1}^{1} \frac{dU}{dY}\frac{dv}{dY} dY + Ha^2 \int_{-1}^{1} UvdY = \int_{-1}^{1} GvdY$$

Since the test function vanishes at the boundaries (due to homogeneous Dirichlet conditions), the boundary terms disappear.

Thus, the weak form of the problem is: Find U ∈ V such that

$$\mathbf{a}(\mathbf{U}, \mathbf{v}) = \mathbf{L}(\mathbf{v}) \ \ \forall\, \mathbf{v} \in \mathbf{V},$$

Where $\mathbf{a}(\mathbf{U}, \mathbf{v}) = \int_{-1}^{1} \frac{dU}{dY}\frac{dv}{dY} dY + Ha^2 \int_{-1}^{1} UvdY$ and $\mathbf{L}(\mathbf{v}) = \int_{-1}^{1} GvdY$.

3.2 Discretization of the Domain

The interval $[-1,1]$ is divided into n finite elements of equal lengthh$_e$.

Each element has two nodes, and linear shape functions are used:

$$N_1(Y) = \frac{Y_2 - Y}{h_e}, N_2(Y) = \frac{Y - Y_1}{h_e}.$$

The approximate solution within each element is written as:

$$U^e(Y) = N_1U_1 + N_2U_2$$

3.3 Element Matrices

Substituting the finite element approximation into the weak form leads to the element equation:

$$[K^e]\{U^e\} = \{F^e\}$$

where the element stiffness matrix is:

$$K^e = \int_{Y_1}^{Y_2} \left[\frac{dN_i}{dY}\frac{dN_j}{dY} + Ha^2N_iN_j\right] dY$$

and the element load vector is:

$$F^e = \int_{Y_1}^{Y_2} [GN_i]dY$$

After evaluating the integrals for linear elements, the element stiffness matrix becomes:

$$K^e = \frac{1}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{Ha^2h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The element load vector is:

$$F^e = \frac{Gh_e}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

3.4 Assembly of Global System

All element matrices are assembled into the global system:

$$[K]\{U\} = \{F\}.$$

The boundary conditions: $U(-1) = 0, U(1) = 0$ are imposed by modifying the global matrix system accordingly.

3.5 Solution Procedure

The resulting system of linear algebraic equations is solved using standard numerical techniques such as Gaussian elimination or LU decomposition. The velocity distribution is obtained for different values of the Hartmann number Ha

3.6 Computational Algorithm

The computational steps are summarized as follows:

1. Discretize the domain into finite elements.
2. Compute element stiffness matrices and load vectors.
3. Assemble global stiffness matrix and load vector.
4. Apply boundary conditions.
5. Solve the linear system.
6. Plot velocity distribution for various Ha values.

4. Results and Discussion

In this section, the numerical results obtained using the Finite Element Method are presented and discussed. The governing boundary value problem was solved for different values of the Hartmann number Hain order to examine the influence of the magnetic field on the velocity distribution within the channel.
For computational purposes, the domain $Y \in [-1,1]$ was discretized into a sufficient number of linear finite elements to ensure convergence of the solution. The pressure gradient parameter G was taken as constant and positive, representing a pressure-driven flow.

4.1 Validation of the Numerical Scheme

For small values of the Hartmann number ($Ha \approx 0$), the governing equation reduces to the classical Poiseuille flow equation. The obtained numerical velocity profile in this limiting case shows the expected parabolic distribution, confirming the correctness and stability of the finite element implementation.
The numerical solution was found to be smooth and symmetric about the channel centerline ($Y = 0$), which is physically consistent with the geometry and boundary conditions.

4.2 Effect of Hartmann Number on Velocity Distribution

The Hartmann number represents the ratio of electromagnetic force to viscous force and plays a crucial role in determining flow behavior. Numerical simulations were carried out for increasing values of Ha, such as $\mathrm{Ha} = 0,2,5,10$.

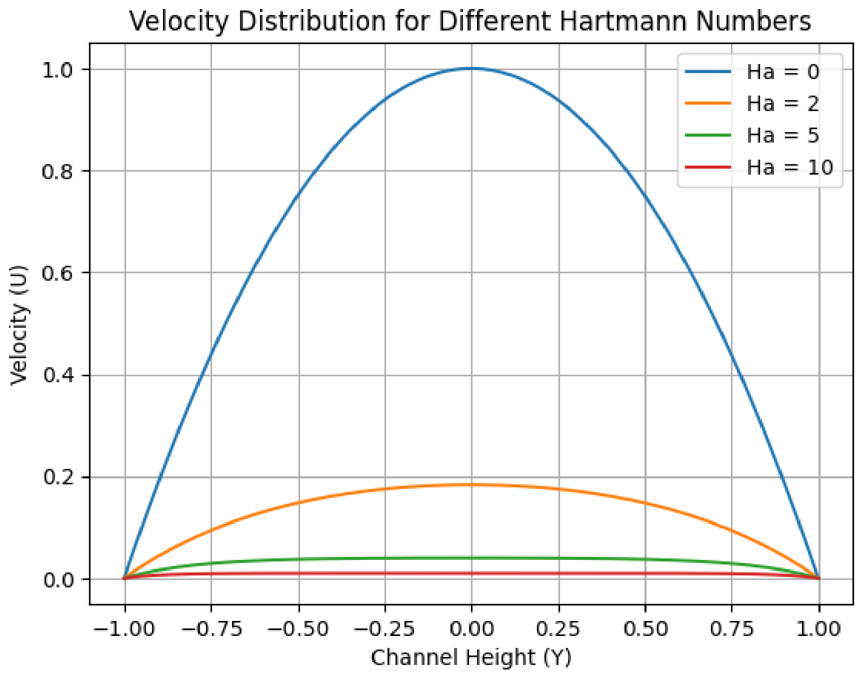

Figure 1: Velocity distribution across the channel for different Hartmann numbers (Ha = 0, 2, 5, 10). Increasing magnetic field strength reduces the velocity and flattens the profile.

The following observations were made:

1. **Reduction in Velocity Magnitude**
   As H_a increases, the maximum velocity at the center of the channel decreases significantly. This is due to the Lorentz force generated by the magnetic field, which opposes fluid motion.
2. **Flattening of Velocity Profile**
   With increasing magnetic field strength, the velocity profile becomes flatter in the core region of the channel. The magnetic force suppresses velocity gradients in the central region.
3. **Boundary Layer Formation**
   For larger Hartmann numbers, thin boundary layers develop near the channel walls. Within these regions, velocity changes rapidly to satisfy the no-slip boundary condition.
4. **Magnetic Damping Effect**
   The applied magnetic field acts as a resistive mechanism, converting kinetic energy into thermal energy. This results in a damping effect on the fluid motion.

4.3 Physical Interpretation

The presence of a transverse magnetic field introduces an additional resistive force proportional to velocity. Mathematically, this appears as the term Ha^2U in the governing equation. As Ha increases, this term becomes dominant compared to the viscous diffusion term.

Physically, the Lorentz force acts against the direction of motion, slowing down the fluid particles. Therefore:

- For $Ha = 0$: Pure viscous flow (parabolic profile)
- For small Ha: Slight reduction in velocity
- For large Ha: Strong suppression of motion and nearly uniform core velocity

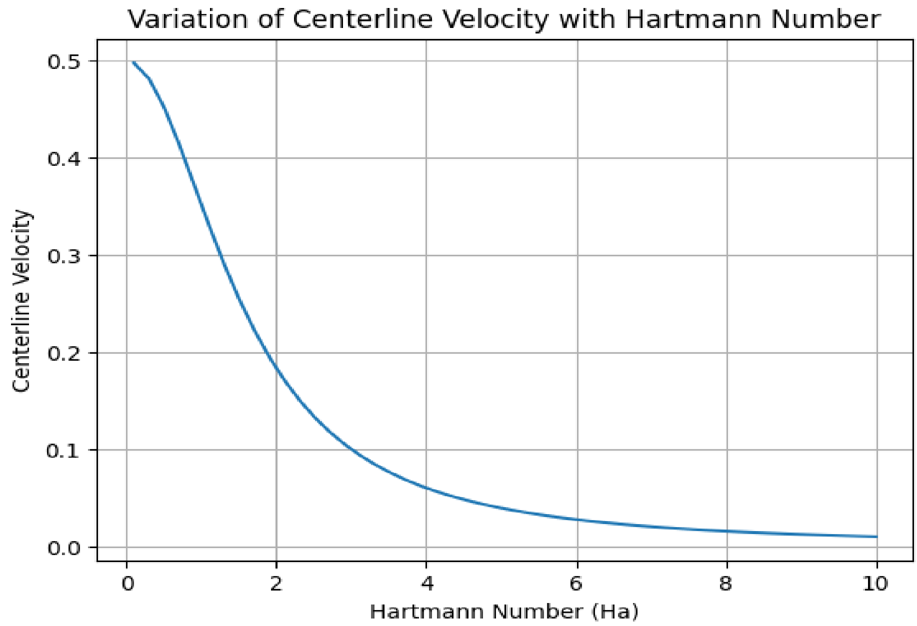

Figure 2: Variation of centerline velocity with Hartmann number. The centerline velocity decreases as the magnetic parameter increases due to the resistive Lorentz force.

This behavior is consistent with classical magnetohydrodynamics theory.

4.4 Convergence and Numerical Stability

The finite element solution shows stable convergence with mesh refinement. Increasing the number of elements improves solution accuracy but does not significantly alter the overall velocity trend, indicating numerical reliability of the method. The stiffness matrix remains symmetric and positive definite, ensuring stable computation for all considered values of the Hartmann number.

4.5 Summary of Findings

The numerical investigation leads to the following key conclusions:

- The magnetic field significantly influences the velocity distribution.
- Increasing Hartmann number reduces the centerline velocity.
- The flow becomes more uniform in the central region.
- The finite element method provides accurate and stable solutions for MHD channel flow problems.

5. Conclusion

The present study investigated the steady magnetohydrodynamic (MHD) flow of an incompressible, electrically conducting viscous fluid in a rectangular channel under the influence of a transverse magnetic field. The governing differential equation was derived from the Navier–Stokes equation by incorporating the Lorentz force term. The resulting boundary value problem was solved numerically using the Finite Element Method (FEM).

A weak formulation of the governing equation was developed, and the computational domain was discretized using linear finite elements. The global stiffness matrix and load vector were assembled systematically, and the resulting system of linear algebraic equations was solved after applying appropriate boundary conditions.

The numerical results clearly demonstrate the significant influence of the magnetic field on the velocity distribution. As the Hartmann number increases, the velocity of the fluid decreases due to the resistive Lorentz force generated by the applied magnetic field. The velocity profile becomes progressively flatter in the central region of the channel, and thin boundary layers are formed near the walls. These findings are consistent with classical MHD flow theory.

The study confirms that the Finite Element Method is an efficient, accurate, and stable numerical technique for solving magnetohydrodynamic flow problems. The method can easily be extended to more complex situations such as unsteady flow, heat transfer, nonlinear magnetic effects, or different geometries.

Overall, this work provides a clear computational framework suitable for academic research and practical engineering applications involving electrically conducting fluids under magnetic influence.

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