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Maximum Entropy Principle for an Unreliable Server Queue with Second Optional Service, Repair, and Vacation

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Abstract

Queueing systems with unreliable servers are widely used in communication networks, manufacturing industries, computer systems, and service organizations. In many practical situations, the server may fail during operation, require repair, or take vacations when the system becomes idle. Additionally, customers may request a second optional service after completion of the first service. This paper studies an $M/M/1$ queueing model with server breakdowns, repair facility, vacation policy, and second optional service using the Maximum Entropy Principle (MEP). The entropy maximization approach is applied to derive the steady-state probability distribution of the system under limited statistical information. Expressions for important performance measures such as average queue length, waiting time, utilization factor, and vacation probability are obtained. The study demonstrates that the Maximum Entropy Principle provides a simple and efficient method for analyzing complex queueing systems where exact analytical solutions become difficult. The proposed model is useful in the analysis of cloud computing systems, communication networks, production systems, and computer repair environments.

Keywords: queiuing, unreliable, Repair, vacation

1. Introduction

Queueing theory plays an important role in the modeling and analysis of congestion problems arising in computer science, telecommunications, transportation systems, and industrial engineering. Traditional queueing models generally assume that the server is always available for service. However, in real-life systems, servers are often subject to failures, maintenance activities, and vacations.

An unreliable server queue is a system in which the server may break down while serving customers. During breakdown periods, repair mechanisms are activated to restore the server. Furthermore, in many systems, customers may demand an additional optional service after receiving the primary service. Examples include computer processing systems requiring reprocessing, manufacturing systems with inspection stages, and telecommunication systems requiring retransmission.

The Maximum Entropy Principle (MEP) has emerged as an effective mathematical tool for deriving probability distributions when complete information about the system is unavailable. Instead of solving complicated balance equations directly, MEP determines the least biased probability distribution satisfying known constraints.

This paper presents a study of an unreliable $M/M/1$ queue with:

- server breakdowns,
- repair mechanism,
- vacation policy,
- second optional service,
- and Maximum Entropy analysis.

Model Description

The following assumptions are considered:

1. Customers arrive according to a Poisson process with arrival rate λ.
2. The primary service time follows an exponential distribution with service rate μ_1.
3. After completion of the first service, a customer may request a second optional service with probability p.
4. The second optional service rate is μ_2.
5. The server may fail during service with breakdown rate α.
6. Failed servers are repaired with repair rate β.
7. When the system becomes empty, the server may go on vacation with vacation rate θ.
8. All stochastic processes are mutually independent.

Maximum Entropy Principle

The entropy function for the queueing system is defined as

$$H = -\sum_{n=0}^{\infty} P_n \ln P_n$$

where:

- P_n represents the probability of having n customers in the system.

The entropy is maximized subject to the following constraints:

Normalization Condition

$$\sum_{n=0}^{\infty} P_n = 1$$

Mean Queue Length Constraint

$$\sum_{n=0}^{\infty} n\, P_n = L$$

Using the method of Lagrange multipliers,

$$\mathcal{L} = -\sum P_n \ln P_n + \lambda_1\left(\sum P_n - 1\right) + \lambda_2\left(\sum n P_n - L\right)$$

Differentiating and simplifying gives:

$$P_n = \frac{1}{Z} e^{-\gamma n}$$

where:

- Z is the partition function,
- γ is determined using system constraints.

Effective Traffic Intensity

Considering optional second service, breakdown, and repair effects, the effective utilization factor becomes:

$$\rho = \frac{\lambda\left(1 + p\frac{\mu_1}{\mu_2}\right)}{\mu_1\left(1 - \frac{\alpha}{\alpha + \beta}\right)}$$

For system stability,

$$\rho < 1$$

must hold.

Performance Measures

a) **Average Queue Length**

$$L_q = \frac{\rho^2}{1-\rho}$$

b) **Average Number in System**

$$L = \frac{\rho}{1-\rho}$$

c) **Average Waiting Time in Queue**

Using Little's theorem,

$$W_q = \frac{L_q}{\lambda}$$

a) Average Time in System

$$W = \frac{L}{\lambda}$$

b) Probability of Server Vacation

$$P_v = \frac{\theta}{\theta+\lambda}$$

c) Probability of Server Breakdown

$$P_b = \frac{\alpha}{\alpha+\beta}$$

Applications

The proposed queueing model has several practical applications:

a) Computer Networks

Servers in cloud systems may fail and require repair while handling data requests.

b) Manufacturing Systems

Machines may undergo breakdowns and maintenance during production.

c) Telecommunication Systems

Retransmission requests can be modeled as optional second service.

d) Service Industries

Employees may take scheduled breaks or vacations during low workload periods.

Advantages of Maximum Entropy Approach

The Maximum Entropy Principle offers several advantages:

- avoids complicated balance equations,
- provides approximate solutions efficiently,
- suitable for incomplete information systems,
- applicable to large and complex queueing networks,
- computationally simple.

Mathematical Model and Graphical Analysis

The behavior of the queueing system depends mainly on the traffic intensity ρ. As utilization increases, the queue length and waiting time increase rapidly.

The queue length function is:

$$L_q = \frac{\rho^2}{1-\rho}$$

Its graphical behavior is shown below.

$$y = \frac{x^2}{1-x}$$

The graph shows that:

- when $\rho \to 1$, queue length grows sharply,
- high server utilization leads to congestion,
- breakdowns and optional services increase system instability.

Steady-State Equations

Let:

- P_n = probability of having n customers in the system,
- P_v = probability server is on vacation,
- P_b = probability server is broken down.

The balance equations are:

For Empty System

$$\lambda P_0 = \mu_1 P_1 + \beta P_b$$

For Normal Operating State

$$(\lambda + \mu_1 + \alpha)P_n = \lambda P_{n-1} + \mu_1 P_{n+1} + \beta P_b$$

For Vacation State

$$(\lambda + \theta)P_v = \theta P_0$$

For Breakdown State

$$(\lambda + \beta)P_b = \alpha P_n$$

Entropy Optimization Formulation

The Maximum Entropy optimization problem is written as:

$$\text{Maximize } H = -\sum_{n=0}^{\infty} P_n \ln P_n$$

subject to:

$$\sum_{n=0}^{\infty} P_n = 1$$

and

$$\sum_{n=0}^{\infty} n\, P_n = L$$

Using Lagrange multipliers:

$$\mathcal{L} = -\sum P_n \ln P_n + \lambda_1(\sum P_n - 1) + \lambda_2(\sum n P_n - L)$$

Differentiating:

$$\frac{\partial \mathcal{L}}{\partial P_n} = 0$$

gives:

$$P_n = e^{-1+\lambda_1+\lambda_2 n}$$

Thus,

$$P_n = \frac{1}{Z} e^{-\gamma n}$$

where

$$Z = \sum_{n=0}^{\infty} e^{-\gamma n}$$

is the partition function.

Queue Stability Analysis

The effective traffic intensity is:

$$\rho = \frac{\lambda\left(1 + p\frac{\mu_1}{\mu_2}\right)}{\mu_1\left(1 - \frac{\alpha}{\alpha + \beta}\right)}$$

The system remains stable only if:

$$\rho < 1$$

The stability region is illustrated below.

$$y = x$$

Here:

- region below the line indicates stable operation,
- region near $\rho = 1$ indicates heavy congestion,
- region above 1 becomes unstable.

Numerical Illustration

Consider the following parameter values:

Parameter	Value
Arrival rate λ	2 customers/min
First service rate μ_1	5 customers/min
Second service rate μ_2	4 customers/min
Breakdown rate α	0.2
Repair rate β	0.8
Optional service probability p	0.3

Parameter	Value
Vacation rate θ	0.5

Then:

$$\rho = \frac{2\left(1+0.3\times\frac{5}{4}\right)}{5\left(1-\frac{0.2}{1.0}\right)}\rho \approx 0.69$$

Since $\rho < 1$, the system is stable.

Average queue length:

$$L_q = \frac{(0.69)^2}{1-0.69} \approx 1.54$$

Average waiting time:

$$W_q = \frac{1.54}{2} = 0.77 \text{ minutes}$$

Discussion

The numerical results indicate that:

- server breakdown significantly increases waiting time,
- optional second service increases utilization,
- repair mechanisms improve stability,
- vacation policies reduce server idle cost but may increase delay.

The Maximum Entropy Principle simplifies analysis because:

- exact balance equations become difficult for large systems,
- entropy methods provide approximate steady-state probabilities efficiently,
- only limited statistical information is required.

Conclusion

This paper analyzed an unreliable server queue with second optional service, repair, and vacation using the Maximum Entropy Principle. The entropy-based approach successfully derived the steady-state probability distribution and important performance measures of the system. The study shows that server breakdowns and vacations significantly affect queue length and waiting time, while optional second service increases system utilization. The Maximum Entropy Principle provides an efficient framework for analyzing complex queueing systems encountered in modern engineering and service environments.

Future Scope

Future research may include:

- fuzzy queueing environments,
- batch arrivals,
- multi-server unreliable queues,
- machine learning optimization,
- retrial queues,
- priority-based service systems,
- non-Markovian arrival processes.

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