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Electrostatic Potential Modeling Using Numerical Methods: Analysis of ADI and Standard Five-Point Schemes

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Abstract

Accurate evaluation of electrostatic potential distribution is essential in diverse fields such as semiconductor device modeling, electromagnetic field analysis, and steady-state heat transfer. This work presents a comparative study of two numerical approaches—the Alternating Direction Implicit (ADI) method and the Standard Five-Point (SFP) finite difference scheme—for solving the two-dimensional Laplace equation on square and rectangular domains with Dirichlet boundary conditions. The ADI method, known for its unconditional stability and efficiency, is implemented in MATLAB and applied to large grid systems, while the SFP scheme provides a direct and straightforward discretization framework. Numerical solutions are validated against analytical results obtained using the separation of variables method. The comparative analysis emphasizes accuracy, convergence characteristics, and computational efficiency. Results indicate that while the SFP method performs reliably for smaller grids, the ADI approach demonstrates faster convergence and superior performance for larger domains. These findings highlight the strengths and limitations of both methods and offer guidance for selecting appropriate numerical techniques in electrostatic potential modeling.

Keywords: Electrostatic potential distribution, Alternating Direction Implicit method, Standard Five-Point scheme, Dirichlet boundary condition, Laplace equation.

1. Introduction

Electrostatic potential distribution is a fundamental concept in physics and engineering, with applications in semiconductor device modeling, electrostatics, heat conduction, hydrodynamics, and stress analysis. The governing equation for such problems is the Laplace equation, a second-order partial differential equation (PDE) that describes steady-state conditions in diverse physical systems. Analytical solutions of the Laplace equation are restricted to simple geometries and boundary conditions, making numerical methods essential for handling complex and realistic domains. A general second-order linear PDE in two variables is given by

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G \qquad 1.1$$

where A, B, C, D, E, F, G are functions of x and y only.

Based on the discriminant $\Delta = B^2 - 4AC$, PDEs are classified as hyperbolic ($\Delta > 0$), elliptic ($\Delta < 0$), or parabolic ($\Delta = 0$). The Laplace equation represents a special case of elliptic PDEs and is directly related to the Poisson equation, which connects electric potential to charge density. In charge-free regions, Poisson's equation reduces to the Laplace equation. For boundary value problems, the Laplace equation is generally associated with well-posed boundary conditions. In this study, Dirichlet boundary conditions are considered. Previous works have addressed numerical solutions of Laplace and Poisson equations using various techniques: Clive [8], Cooper [9], and Adak [1–6] have contributed to the development of finite difference methods and PDE theory; Patil and Prasad [19], Morales et al. [14], and Li et al. [13] discussed approximate solutions of the two-dimensional Laplace equation with Dirichlet conditions; Dhumal and Kiwne [11], and Ubaidullah and Muhammad [20] also contributed to this area. Extensions to Poisson's equation include the works of Benyam and Purnachandra [7] and Pandey and Jaboob [18]. Other notable contributions include Eyaya [12] on the finite volume method, Dambroshe [10] on three-dimensional numerical algorithms, and Mostafa et al. [15–17] on anisotropic potential using finite element methods. Among numerical approaches, the Finite Difference Method (FDM) remains popular due to its simplicity. The Standard Five-Point (SFP) scheme, a classical FDM discretization, provides accurate solutions but becomes computationally intensive for larger domains. To improve efficiency, iterative approaches such as the Alternating Direction Implicit (ADI) method have been introduced. The ADI scheme is unconditionally stable and reduces multidimensional problems to a sequence of one-dimensional tridiagonal systems, lowering computational effort while preserving accuracy.

The objective of this work is to perform a comparative analysis of the ADI and SFP methods for solving the two-dimensional Laplace equation on square and rectangular domains with Dirichlet boundary conditions. Numerical solutions are obtained using MATLAB and validated against analytical solutions derived by the separation of variables. The comparison emphasizes accuracy, convergence behavior, and computational efficiency, highlighting the strengths and limitations of both approaches for electrostatic potential distribution problems.

2. **Mathematical Formulation**

The Laplace equation with two spatial dimensions is given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad 0 \le x \le a, 0 \le y \le b \qquad 2.1$$

is a canonical elliptic PDE. It arises in electrostatics as the special case of Poisson's equation, when charge density vanishes. Boundary conditions are essential for uniqueness; here, Dirichlet boundary conditions are imposed, where the potential is prescribed on domain boundaries.

3. NUMERICAL METHODS

We consider Laplace's equation in two dimensions, viz.,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad 3.1$$

3.1 Alternating Direction Implicit (ADI) Method

The ADI method is an iterative approach that alternates between implicit steps in the x- and y-directions. Each iteration involves solving tridiagonal systems, which are computationally efficient. The scheme is unconditionally stable and converges faster compared to explicit finite difference methods. Its efficiency makes it suitable for large problem domains.

Using the central difference approximation for the second-order derivatives in Equation (3.1), we obtain:

$$\frac{T_{i-1,j}-2T_{i,j}+T_{i+1,j}}{h^2}+\frac{T_{i,j-1}-2T_{i,j}+T_{i,j+1}}{k^2}=0 \qquad 3.2$$

If the mesh sizes in the x- and y-directions are the same, i.e., h=k, Equation (3.2) reduces to:

$$T_{i+1,j}+T_{i-1,j}+T_{i,j+1}+T_{i,j-1}-4\ T_{i,j}=0 \qquad 3.3$$

Equation (3.3) is known as the Standard Five-Point (SFP) formula, which implies that the temperature (or potential) at any interior grid point is equal to the average of its four nearest neighbors (Figure 1).

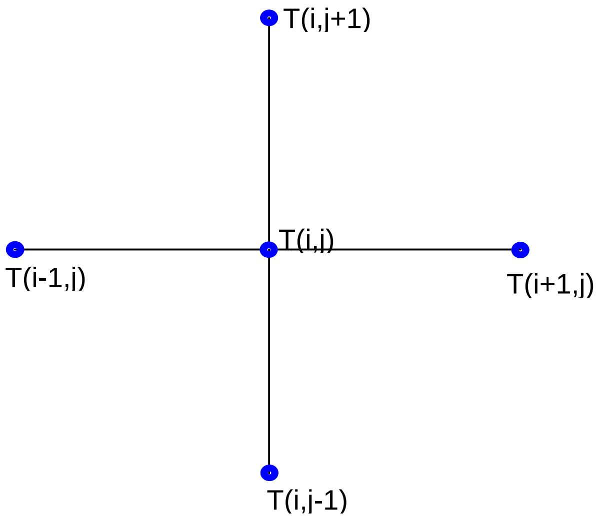

Figure 1 Temperature at any point average of surrounding points

Equation (3.3) can be rearranged in two equivalent ways:

$$T_{i-1,j}-4\ T_{i,j}+T_{i+1,j}=-T_{i,j-1}-T_{i,j+1} \qquad 3.4$$

Or

$$T_{i,j-1}-4\ T_{i,j}+T_{i,j+1}=-T_{i-1,j}-T_{i+1,j} \qquad 3.5$$

These rearrangements form the basis of the Alternating Direction Implicit (ADI) method, which is an iterative scheme. The iteration formulas are written as:

$$T_{i-1,j}^{n+1}-4T_{i,j}^{n+1}+T_{i+1,j}^{n+1}=-T_{i,j-1}^{n}-T_{i,j+1}^{n} \qquad 3.6$$

and

$$T_{i,j-1}^{n+2}-4T_{i,j}^{n+2}+T_{i,j+1}^{n+2}=-T_{i-1,j}^{n+1}-T_{i+1,j}^{n+1} \qquad 3.7$$

Here, Equation (3.6) is applied along rows (j=1,2,…,n−1), while Equation (3.7) is applied along columns (i=1,2,…,n−1). At each stage, the scheme produces a tridiagonal system of equations, which can be solved efficiently using the Thomas algorithm.

For example, for the first row (j=1), Equation (3.6) becomes:

$$T_{i-1,1}^{n+1} - 4T_{i,1}^{n+1} + T_{i+1,1}^{n+1} = -T_{i,0}^{n} - T_{i,2}^{n} \qquad (i = 1, 2, 3, \dots \dots, n-1) \qquad 3.8$$

With the given boundary conditions, Equation (3.8) defines a tridiagonal system that is solved to obtain $T_{i,1}^{n+1}$. The process is then repeated for
j=2,3,…,n−1. In the next half-step, Equation (3.7) is applied along the columns, yielding updated values $T_{i,j}^{n+2}$. Thus, in the ADI method, Equations (3.6) and (3.7) are used alternately, ensuring efficient convergence. At each iteration step, only tridiagonal systems are solved, making the method computationally attractive for large domains.
The stopping criterion for the iteration is

$$\varepsilon_{ij} = \left| \frac{T_{i,j\,present} - T_{i,j\,previous}}{T_{i,j\,present}} \right| < 10^{-n}$$

This ensures that the solution has reached the desired level of accuracy while satisfying both the Laplace equation and the imposed boundary conditions.

3.2 Standard Five Point (SFP) method

The Finite Difference Method (FDM) replaces the governing differential equation with an approximately equivalent finite difference equation. The Standard Five-Point (SFP) scheme is one of the most widely used finite difference approaches for solving the two-dimensional Laplace equation. The procedure for obtaining the numerical solution involves the following steps:

(i) Discretize the solution domain using grids and nodes into small rectangles.
(ii) Approximate the given differential equation by equivalent finite difference equations that give the solutions to the grid points.
(iii) Using the initial boundary condition difference equation produce the system of simultaneous equation.
(iv) Solve the system of equation using Gauss Seidel iteration method.

To find the solution of the function $T(x, y)$ on the region R, we divide the region on XY plane using vertical and horizontal lines into equal rectangles or meshes of sides h and k along x and y direction respectively such that $x = ih,\ \ i = 0, 1, 2 \dots \dots$ and
$y = jh,\ \ j = 0, 1, 2 \dots \dots$
Derivatives in Laplace / Poisson equation are approximated by Taylor's series expansion

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{h^2} + O(h^2) \qquad 3.9$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{T_{i,j-1} - 2T_{i,j} + T_{i,j+1}}{k^2} + O(k^2) \qquad 3.10$$

The terms $O(h^2)$ and $O(k^2)$ denote the order of local truncation error and is also known as the order of method. After neglecting the truncation, error and using equations 3.1 in equation 3.9 and 3.10, we obtain the standard five point finite difference formulae

$$T_{i,j} = \frac{1}{4}\left(T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}\right) \qquad 3.11$$

3.3 Separation Variable Method

Separation of variables is used to reduce partial differential equation to ordinary differential equations. In the method of separation variables, the solution of Laplace equation is considered as a product of two individual functions as

$T(x,y) = X(x)Y(y)$.

Laplace equation has been satisfied by above solution. Then equation (2.1) reduced to

$\frac{X''}{X} = -\frac{Y''}{Y} = -p^2$ (P is positive constant)

Therefore the exact solution of Laplace equation is

$$T(x,y) = (A\cos \mathrm{px} + \mathrm{B}\sin \mathrm{px})(\mathrm{C}\cos \mathrm{hpy} + \mathrm{D}\sin \mathrm{hpy})$$

Using Dirichlet Boundary condition defined in figure 2, we finally obtain exact solution of Laplace equation $T(x,y) = \frac{T_0}{\sin h\frac{\pi b}{a}} \sin\frac{\pi x}{a} \sin h\frac{\pi b}{a}$ 3.12

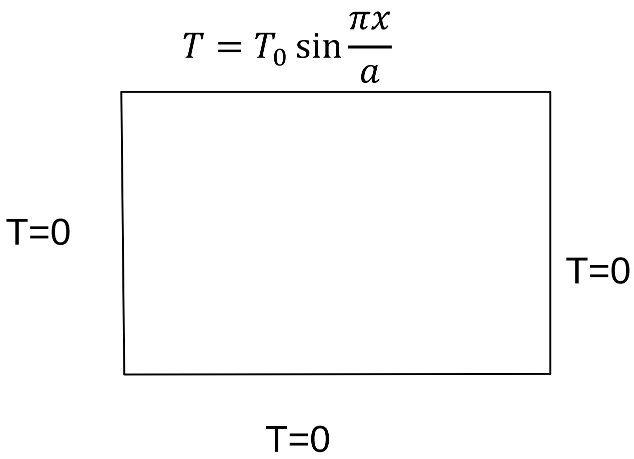

Figure 2 Dirichlet Boundary conditions are specified in a rectangular domain.

4. Numerical Solution by Using Adi Method

Problem 1: (Rectangular domain)

Consider Laplace equation $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0,\ 0 \le x \le 5,\ 0 \le y \le 3$, for calculating potential energy distribution along the boundaries stress are defined by $T(0,y) = T(5,y) = 0, T(x,0) = 0$ and $T(x,3) = x(5-x)$

Solution: Here mesh size $h = 1$, 8 interior points (unknown) in rectangular domain shown in figure 3. We apply formulae given in equations (3.6) and (3.7) for calculating stress at interior points in rectangular domain. To start the iterations, we set $n = 0$ (1^{st} iteration, let $m = 1$) for the first row, $j = 1$ for $m = 1$, then, equation (3.6) gives

$$T_{i-1,1}^{(1)} - 4T_{i,1}^{(1)} + T_{i+1,1}^{(1)} = -T_{i,0}^{(0)} - T_{i,2}^{(0)}$$

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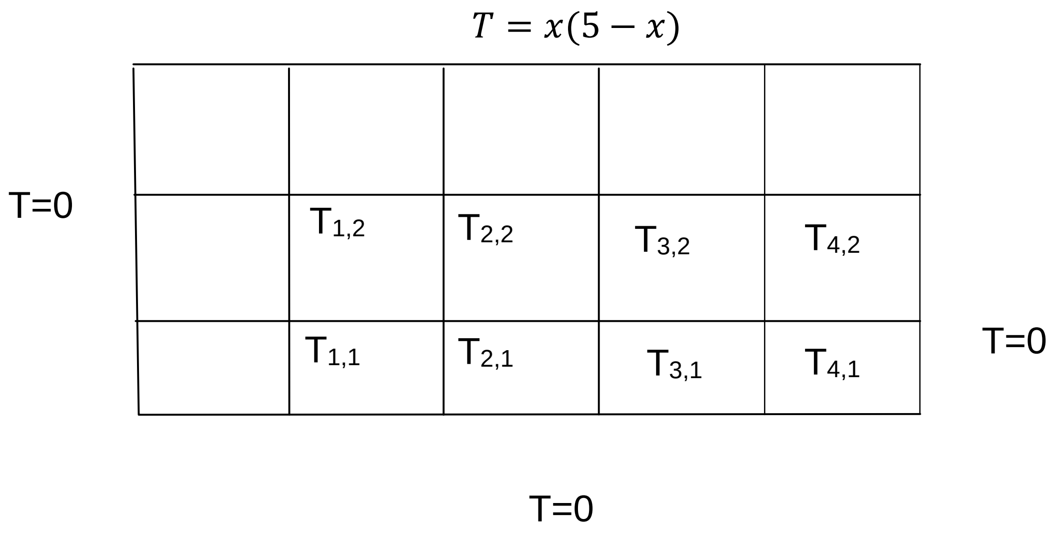

with $i = 1,2,3,4$ this gives four simultaneous equations

$$T_{0,1}^{(1)} - 4T_{1,1}^{(1)} + T_{2,1}^{(1)} = -T_{1,0}^{(0)} - T_{1,2}^{(0)}$$

$$T_{1,1}^{(1)} - 4T_{2,1}^{(1)} + T_{3,1}^{(1)} = -T_{2,0}^{(0)} - T_{2,2}^{(0)}$$

$$T_{2,1}^{(1)} - 4T_{3,1}^{(1)} + T_{4,1}^{(1)} = -T_{3,0}^{(0)} - T_{3,2}^{(0)}$$

$$T_{3,1}^{(1)} - 4T_{4,1}^{(1)} + T_{5,1}^{(1)} = -T_{4,0}^{(0)} - T_{4,2}^{(0)}$$

Substituting the boundary values in above simultaneous equation, we obtain tridiagonal matrix form

$$\begin{pmatrix} -4 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & -4 \end{pmatrix} \begin{pmatrix} T_{1,1} \\ T_{2,1} \\ T_{3,1} \\ T_{4,1} \end{pmatrix} = \begin{pmatrix} T_{1,2}^{0} \\ T_{2,2}^{0} \\ T_{3,2}^{0} \\ T_{4,2}^{0} \end{pmatrix}$$

- $\begin{pmatrix} -4 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & -4 \end{pmatrix} \begin{pmatrix} T_{1,1} \\ T_{2,1} \\ T_{3,1} \\ T_{4,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$
- $T_{1,1} = 0, T_{2,1} = 0,\ T_{3,1} = 0, T_{4,1} = 0,$ (along first row).

Again, we set $n = 0$, for the second row, $j = 2$. then, equation (3.6) gives with

$i = 1,2,3,4$ this gives five simultaneous equations

$$T_{0,2}^{(1)} - 4T_{1,2}^{(1)} + T_{2,2}^{(1)} = -T_{1,1}^{(0)} - T_{1,3}^{(0)}$$

$$T_{1,2}^{(1)} - 4T_{2,2}^{(1)} + T_{3,2}^{(1)} = -T_{2,1}^{(0)} - T_{2,3}^{(0)}$$

$$T_{2,2}^{(1)} - 4T_{3,2}^{(1)} + T_{4,2}^{(1)} = -T_{3,1}^{(0)} - T_{3,3}^{(0)}$$

$$T_{3,2}^{(1)} - 4T_{4,2}^{(1)} + T_{5,2}^{(1)} = -T_{4,1}^{(0)} - T_{4,3}^{(0)}$$

Substituting the boundary values in above simultaneous equation, we obtain tridiagonal matrix form

$$\begin{pmatrix} -4 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & -4 \end{pmatrix}\begin{pmatrix} T_{1,2} \\ T_{2,2} \\ T_{3,2} \\ T_{4,2} \end{pmatrix} = \begin{pmatrix} T_{1,3}^{0} \\ T_{2,3}^{0} \\ T_{3,3}^{0} \\ T_{4,3}^{0} \end{pmatrix}$$

➢ $$\begin{pmatrix} -4 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & -4 \end{pmatrix}\begin{pmatrix} T_{1,2} \\ T_{2,2} \\ T_{3,2} \\ T_{4,2} \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 6 \\ 4 \end{pmatrix}$$

- ➢ $T_{1,2} = -1.636363$, $T_{2,2} = -2.545454$, $T_{3,2} = -2.545454$, $T_{4,2} = -1.636363$, (along second row).

Having completed the computations on two rows, we now alternate the direction and compute the function values on the columns, starting with the first one. For this, we use equation (3.7) with $n = 0$. Setting $i = 1$ for $m = 2$ equation (3.7) becomes

$$T_{1,j-1}^{(2)} - 4T_{1,j}^{(2)} + T_{1,j+1}^{(2)} = -T_{0,j}^{(1)} - T_{2,j}^{(1)}$$

Putting $j = 1$ and $j = 2$ in the above (for column wise calculation), we obtain the equations

$$T_{1,0}^{(2)} - 4T_{1,1}^{(2)} + T_{1,2}^{(2)} = -T_{0,1}^{(1)} - T_{2,1}^{(1)}$$

And

$$T_{1,1}^{(2)} - 4T_{1,2}^{(2)} + T_{1,3}^{(2)} = -T_{0,2}^{(1)} - T_{2,2}^{(1)}$$

Substituting the boundary values we obtain,

$$-4T_{1,1}^{(2)} + T_{1,2}^{(2)} = 0$$

$$T_{1,1}^{(2)} - 4T_{1,2}^{(2)} + 4 = -2.545454$$

Solving these equations we get,

$T_{1,1}^{(2)} = 0.43636$ and $T_{1,2}^{(2)} = 1.74544$

Similarly, along second column, $T_{2,1}^{(2)} = 0.6788$ and $T_{2,2}^{(2)} = 2.71515$

Along third column, $T_{3,1}^{(2)} = 0.6788$ and $T_{3,2}^{(2)} = 2.71515$

Along fourth column, $T_{4,1}^{(2)} = 0.43636$ and $T_{4,2}^{(2)} = 1.74544$

The iteration are continued to improve the function values obtained first on the rows, then on the column, and so on for $m = 3, 4, 5, \dots \dots \dots 20$.

Table 1 ADI iterative process for Laplace Equation with **Dirichlet** boundary condition in a **rectangular** domain

Iterations	$T_{1,1}$	$T_{2,1}$	$T_{3,1}$	$T_{4,1}$	$T_{1,2}$	$T_{2,2}$	$T_{3,2}$	$T_{4,2}$
1	0	0	0	0	1.6364	2.5454	2.5454	1.6364
2	0.4364	0.6788	0.6788	0.4364	1.7454	2.7152	2.7152	1.7454
3	0.7229	1.1460	1.1460	0.7229	1.8171	2.8319	2.8319	1.8171
4	0.7611	1.2083	1.2083	0.7611	1.8983	2.9643	2.9643	1.8983
5	0.7872	1.2505	1.2505	0.7872	1.9538	3.0540	3.0540	1.9538
6	0.8037	1.277	1.277	0.8037	1.9644	3.0712	3.0712	1.9644
7	0.8150	1.2954	1.2954	0.8150	1.9717	3.0829	3.0829	1.9717
8	0.8177	1.2997	1.2997	0.8177	1.9752	3.0886	3.0886	1.9752
9	0.8194	1.3027	1.3027	0.8194	1.9775	3.0924	3.0924	1.9775
10	0.8202	1.3039	1.3039	0.8202	1.9782	3.0934	3.0934	1.9782
11	0.8207	1.3047	1.3047	0.8207	1.9786	3.0942	3.0942	1.9786
12	0.8209	1.3049	1.3049	0.8209	1.9787	3.0944	3.0944	1.9787

In this problem the obtained results are also verified using a MATLAB program, where we do iterations up to m = 20, the results were found to converge at m = 12 up to three decimal points shown in Table1. For different values of m that means at different step of iteration numerical results are showing in table 2.

Table 2 Results for $m = 1,\ m = 5,\ m = 10,\ m = 15$

T_{ij}	$m = 1$	$m = 5$	$m = 10$	$m = 15$
$T_{1,1}$	0.436363636363636	0.803737206778529	0.820720986111890	0.821043224903156
$T_{1,2}$	1.745454545454545	1.964440437432173	1.978593955318144	1.978937831729706
$T_{2,1}$	0.678787878787879	1.277239613768539	1.304726541885639	1.305247937574598
$T_{2,2}$	2.715151515151515	3.071259147953363	3.094165005071496	3.094721411415559
$T_{3,1}$	0.678787878787879	1.277239613768539	1.304726541885639	1.305247937574598
$T_{3,2}$	2.715151515151515	3.071259147953363	3.094165005071496	3.094721411415559
$T_{4,1}$	0.436363636363636	0.803737206778529	0.820720986111890	0.821043224903156
$T_{4,2}$	1.745454545454545	1.964440437432173	1.978593955318144	1.978937831729706

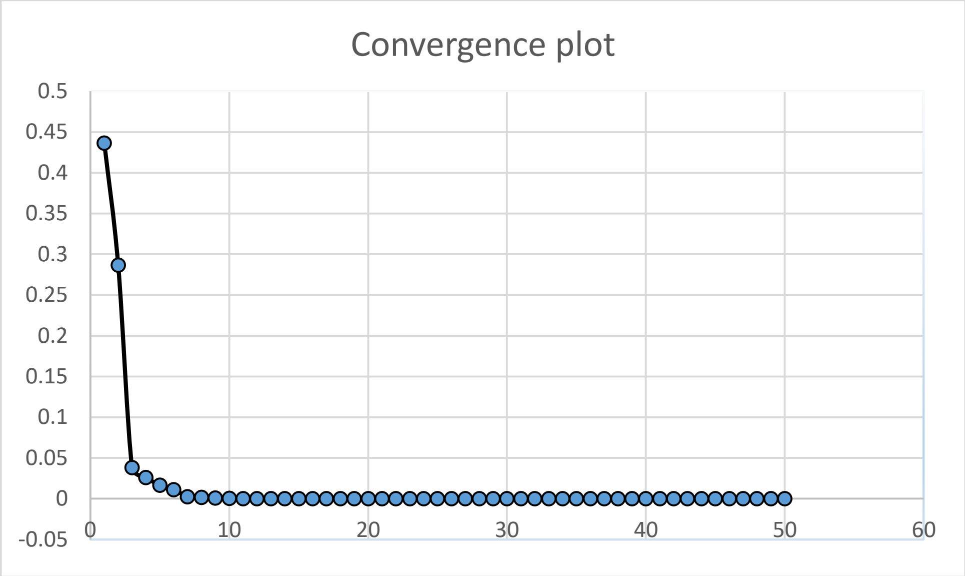

Figure 4 Convergence plot of ADI method for problem 1

Figure 4 showing the convergence plot for problem 1 using ADI method.

SFP Method

Problem 1 has been solved by using standard five point (SFP) method as explained in equation 3.11, mesh points are shown in figure 1, we have

$$\begin{pmatrix} -4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -4 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -4 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -4 \end{pmatrix} \begin{pmatrix} T_{1,1} \\ T_{1,2} \\ T_{2,1} \\ T_{2,2} \\ T_{3,1} \\ T_{3,2} \\ T_{4,1} \\ T_{4,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1(5-1) \\ 0 \\ 2(5-2) \\ 0 \\ 3(5-3) \\ 0 \\ 4(5-4) \end{pmatrix}$$

Solving simultaneously, that gives the following results

- $T_{1,1} = 0.821043462$ $T_{1,2} = 1.978939842$
- $T_{2,1} = 1.305247147$ $T_{2,2} = 3.094721813$
- $T_{3,1} = 1.305247147$ $T_{3,2} = 3.094721813$
- $T_{4,1} = 0.821043462$ $T_{4,2} = 1.978939842$

Using separation of variables method we obtain the exact solution of the Laplace equation with Dirichlet condition (problem 1) that is final solution to the stress distribution is given by

$$T(x,y) = \frac{200}{\pi^3 \sin h\frac{3\pi}{5}} \sin\frac{\pi x}{5} \sin h\ \frac{\pi y}{5}$$

This equation will give the exact solution at each interior point of rectangular domain defined as figure 2. Comparison of numerical results with exact results and error are calculated which is mentioned in table 3.

Table 3 Comparison with Exact Solution, ADI and SFP solution

Nodes	ADI solution	SFP solution	Exact solution
$T_{1,1}$	0.821043224903156	0.821043462	0.790171122323631
$T_{2,1}$	1.305135272972097	1.305247147	1.278523732848285
$T_{3,1}$	1.305135272972097	1.305247147	1.278523732848285
$T_{4,1}$	0.820973594279218	0.821043462	0.790171122323631
$T_{1,2}$	1.978871288425587	1.978939842	1.902687954157417
$T_{2,2}$	3.094613742194746	3.094721813	3.078613779811702
$T_{3,2}$	3.094613742194746	3.094721813	3.078613779811702
$T_{4,2}$	1.978871288425587	1.978939842	1.902687954157417

Problem 2 (Square Domain)

Potential energy distribution problem defined in square domain which satisfies Laplace equation $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0,\ \ 0 \le x, y \le 4,$ due to boundary conditions $T(0,y) = T(4,y) = 0,$ $T(x,0) = 0$ and $T(x,4) = \frac{2}{5} sin\frac{\pi x}{4}$ using ADI method.

Solution: Here $h = 1,$ 9 interior points (unknown) in square domain shown in figure 5.

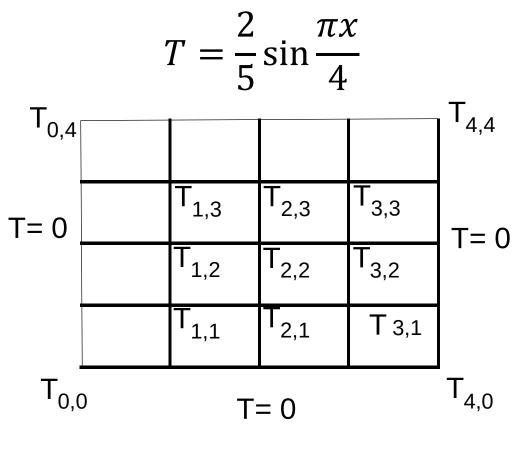

Figure 5 Grid points with boundary condition in a square domain for problem 2

Using equations 3.6 and 3.7 we obtain the approximate solution at each node. ADI iterative process for numerical solution is shown in Table 4.

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Table 4 Potential energy distribution in square domain with Dirichlet boundary condition.

Iterations	$T_{1,1}$	$T_{2,1}$	$T_{3,1}$	$T_{1,2}$	$T_{2,2}$	$T_{3,2}$	$T_{1,3}$	$T_{2,3}$	$T_{3,3}$
1	0	0	0	0	0	0	0.10938	0.15469	0.10938
2	0.00781	0.01105	0.00781	0.03125	0.044198	0.03125	0.11720	0.16574	0.11720
3	0.01209	0.01709	0.01209	0.04835	0.06837	0.04834	0.04835	0.17178	0.12147
4	0.01758	0.02486	0.01758	0.05323	0.07528	0.05323	0.12696	0.17955	0.12696
5	0.02059	0.02911	0.02058	0.05590	0.07905	0.05590	0.12997	0.18380	0.12997
6	0.02177	0.03080	0.02177	0.05799	0.08202	0.05799	0.13116	0.18549	0.13116
7	0.02243	0.03172	0.02243	0.05914	0.08364	0.05914	0.13181	0.18641	0.13181
8	0.02285	0.03231	0.02285	0.05968	0.08440	0.05968	0.13223	0.18700	0.13223
9	0.02308	0.03264	0.02308	0.05997	0.08481	0.05997	0.13246	0.18733	0.13246
10	0.02319	0.03280	0.02319	0.06015	0.08506	0.06015	0.13258	0.18749	0.13258
11	0.02326	0.03289	0.02326	0.06024	0.08519	0.06024	0.13264	0.18759	0.13264
12	0.02329	0.03294	0.02329	0.06029	0.08526	0.06029	0.13268	0.18763	0.13268

Eighten iterations are necessary to reach a solution of Laplace equation with Dirichlet condition in square domain. The obtained results are also verified using a MATLAB program, where we do iterations up to $m = 20$ (21 iteration). For different values of m that means at different step of iteration numerical results are showing in table 5.

Table 5 Results for $m = 1,\ m = 6,\ m = 12,\ m = 18$

T_{ij}	$m = 1$	$m = 6$	$m = 12$	$m = 18$
$T_{1,1}$	0.007813116	0.02242957	0.02331763	0.023340559
$T_{1,2}$	0.031252466	0.05914584	0.06032178	0.060354444
$T_{1,3}$	0.117196748	0.13181320	0.13270126	0.132724191
$T_{2,1}$	0.011049415	0.03172020	0.03297611	0.033008535
$T_{2,2}$	0.044197661	0.08364485	0.08530788	0.085354074
$T_{2,3}$	0.165741231	0.18641202	0.18766792	0.187700351
$T_{3,1}$	0.007813116	0.02242957	0.02331763	0.023340559
$T_{3,2}$	0.031252466	0.05914584	0.06032178	0.060354444
$T_{3,3}$	0.117196748	0.13181320	0.13270126	0.132724191

Using separation of variables method (equation 3.9) we obtain the exact solution of the Laplace equation with Dirichlet condition (problem 2) that is final solution to the stress distribution is

$$T(x,y) = \frac{2}{5\sin h\,\pi}\sin\frac{\pi x}{4}\sin h\ \frac{\pi y}{4}$$

This equation will give the exact solution at each interior point of square domain defined as figure 5. Comparison of numerical results with exact results and error are calculated which is mentioned in table 6.

Table 6 Comparison with Exact Solution and ADI Numerical solution

Nodes	ADI Numerical solution	Exact solution	Error
$T_{1,1}$	0.023340559	0.02127481	0.0020657

$T_{1,2}$	0.060354444	0.05636161	0.0039928
$T_{1,3}$	0.132724191	0.12803940	0.0046847
$T_{2,1}$	0.033008535	0.03008712	0.0029214
$T_{2,2}$	0.085354074	0.07970736	0.0056467
$T_{2,3}$	0.187700351	0.1810750	0.0066252
$T_{3,1}$	0.023340559	0.02127481	0.0020657
$T_{3,2}$	0.060354444	0.05636161	0.0039928
$T_{3,3}$	0.132724191	0.12803940	0.0046847

5. Conclusion

This work presented a comparative study of the Alternating Direction Implicit (ADI) method and the Standard Five-Point (SFP) finite difference scheme for solving the two-dimensional Laplace equation with Dirichlet boundary conditions. Numerical experiments on square and rectangular domains demonstrated that both approaches yield results in good agreement with the analytical solution obtained through separation of variables. The study confirms that while the SFP method is simple and effective for smaller grid sizes, its efficiency decreases with finer meshes. In contrast, the ADI method exhibits unconditional stability, faster convergence, and superior accuracy, particularly for large-scale problems. These characteristics make the ADI approach more suitable for practical applications where computational efficiency and accuracy are critical. The accuracy of both methods can be further improved by refining the mesh, though this comes at the expense of higher computational cost. Future extensions of this work may include applying the ADI method to irregular or complex geometries and exploring its applicability to coupled problems such as electrostatic potential, stress analysis, and heat transfer in multidimensional domains.

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