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Distributional Aboodh-Stieltjes Transform and its Analytical Structure

A. N. Rangari[1], S. R. Khandekar[2], V. A. Sharma[3]

[1] Department of Mathematics, Adarsha College, Dhamangaon Rly (M.S.), Maharashtra, India, Affiliated to Sant Gadge Baba Amravati University, Amravati
[2,3] Department of Mathematics, Arts, Commerce and Science College, Amravati, Maharashtra, India, Affiliated to Sant Gadge Baba Amravati University, Amravati

Abstract
This paper introduces a generalization of the Aboodh–Stieltjes transform in the distributional sense. Suitable testing function spaces are constructed using the Gelfand–Shilov technique to provide a rigorous framework for defining the transform on distributions. Fundamental properties of the generalized Aboodh–Stieltjes transform are investigated, and an analyticity theorem is established to describe its analytical behavior in the transform domain. The proposed approach extends the classical Aboodh–Stieltjes transform and provides a useful framework for the analysis of generalized functions and related operational problems.

Keywords: Aboodh Transform, Stieltjes Transform, Aboodh–Stieltjes Transform, Testing Function Space, Generalized function, Analyticity Theorem.

1. Introduction

In mathematical sciences, a transformation is a mathematical tool used to convert a function f(z) into another function in a new variable, generally denoted by F(s). Integral transforms have been successfully used for nearly two centuries in solving various problems arising in applied mathematics, mathematical physics, and engineering sciences [1].

The Aboodh transform is an integral transform introduced by Khalid Aboodh in 2013. It was derived from the classical Fourier integral with the aim of simplifying the process of solving ordinary and partial differential equations in the time domain [2].

T. J. Stieltjes (1856–1894) introduced the Stieltjes transform during his studies on continued fractions. The Stieltjes transform is a generalization of the Riemann integral that allows integration with respect to a function of bounded variation and has important applications in functional analysis [3], [4].
In the present work, we combine the Aboodh transform and the Stieltjes transform with the help of a suitable kernel to formulate a new integral transform, called the Aboodh–Stieltjes Transform. This paper introduces the Aboodh–Stieltjes transform together with its definition and analytical structure in the generalized distributional sense [5].

The paper is organized as follows:

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1. Definition of the Aboodh–Stieltjes Transform.
2. Testing Function Spaces.
3. Definition of Distributional Generalized Aboodh–Stieltjes Transform.
4. Proof of the Analyticity Theorem.
5. Conclusion.

The notations and terminologies used throughout this paper are adopted from Zemanian [6], [7].

2. Definitions

The Aboodh Transform with parameter ‘s’ of the function *f(t)* is denoted by *A[f(t)] = F(s)* and is given by

$A[f(t)] = F(s) = \frac{1}{s}\int_0^{\infty}(e)^{-st} f(t)dt$ for a parameter *s > 0*. (2.1)

The Stieltjes Transform with parameter ‘*y*’ of the function *f(x)* is denoted by

$S[f(x)] = F(y) = \int_0^{\infty}(x+y)^{-p} f(x)dx$ for parameter *y > 0*. (2.2)

The conventional Aboodh- Stieltjes Transform is defined as

$AS\{f(t,x)\} = F(s,y) = \frac{1}{s}\int_0^{\infty}\int_0^{\infty} e^{-st}(x+y)^{-p} f(t,x)dtdx$ (2.3)

Where, $K(t,x) = \frac{1}{s}e^{-st}(x+y)^{-p}$

3. Testing Function Spaces

3.1 *The Space* $AS_{\propto}$:

Let I be the open set in $R_+ \times R_+$ and E_+ denotes the class of infinitely differentiable function defined on I, then the space AS_α is given by

$$AS_{\propto} = \{\emptyset : \emptyset \in E_+ | \Upsilon_{a,b,p,l,q}\emptyset(t,x) = \sup_{I_1}|K_{a,b}(t)(1+x)^p D_t^l (xD_x)^q \emptyset(t,x)| \leq \leq C_{plq}A^a a^{a\propto}\}$$

For each p, l, q = 0,1,2,3,…

Where the constants A and C_{lq} depends on the testing function $\emptyset(t, x)$.

Also where, $K_{a,b}(t) = \begin{cases} e^{at}, & 0 \leq t < \infty \\ e^{bt}, & -\infty < t < 0 \end{cases}$

3.2 *The Space* AS_{β} :

Let I be the open set in $R_+ \times R_+$ and E_+ denotes the class of infinitely differentiable function defined on I, then the space AS_β is given by

$$AS_{\beta} = \{\emptyset : \emptyset \in E_+ | \sigma_{a,b,p,l,q}\emptyset(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<\infty}}|K_{a,b}(t)(1+x)^p D_t^l (xD_x)^q \emptyset(t,x)| \leq C_{apq}B^l l^{l\beta}\}$$

For each a, p, q = 0,1,2,3,…

Where the constants B and C_{apq} depends on the testing function $\emptyset(t, x)$.

3.3 *The Space* AS_{α}^{β}

Let I be the open set in $R_+ \times R_+$ and E_+ denotes the class of infinitely differentiable function defined on I, then the space AS_{α}^{β} is given by,

$$AS_{\alpha}^{\beta} = \{\emptyset : \emptyset \in E_+ | \rho_{a,b,p,l,q}\emptyset(t,x)$$
$$= \sup_{I_1}|K_{a,b}(t)(1+x)^p D_t^l (xD_x)^q \emptyset(t,x)| \leq CA^a a^{a\propto} B^l l^{l\beta}\}$$

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where a, l = 1,2,3,…
Also where the constants A, B and C depends on the testing function Ø(t, x).
This space is the combination of above two spaces.

4. Distributional Generalized Aboodh-Stieltjes Transform

For $f(t,x) \in AS_{\alpha}^{*\beta}$, where $AS_{\alpha}^{*\beta}$ is the dual space of AS_{α}^{β}. It contains all distributions of compact support and let $\emptyset(t,x) \in AS_{\alpha}^{\beta}$.
Then the Distributional Aboodh-Stieltjes Transform is a function of $f(t,x)$ is defined as

$$AS\{f(t,x)\} = F(s,y) = \langle f(t,x), \frac{1}{s}\, e^{-st}(x+y)^{-p} \rangle \tag{4.1}$$

Where for each fixed *t (0 < t < ∞)*, and *x (0 < x < ∞)*, *s > 0*, *p > 0*, the right-hand side of above equation (4.1) has a sense as an application of $f(t,x) \in AS_{\alpha}^{*\beta}$ to
$\frac{1}{s}\, e^{-st}(x+y)^{-p} \in AS_{\alpha}^{\beta}$.

5. Analyticity of Distributional Aboodh-Stieltjes Transform

5.1 Theorem:
Let $f(t,x) \in AS_{\alpha}^{*\beta}$ and it's Aboodh-Stieltjes Transform F(s, y) is defined as

$$AS\{f(t,x)\} = F(s,y) = \langle f(t,x), \frac{1}{s}\, e^{-st}(x+y)^{-p} \rangle$$

Then *F(s, y)* is analytic for some fixed *s > 0*, *y > 0* on Ω_f where,
$\Omega_f = \{(s,y): \sigma_1 < s < \sigma_2\}$ and $\Omega_f = \{(s,y): \sigma_1 < y < \sigma_2\}$
Then

(i) $$D_s F(s,y) = \langle f(t,x), \left\{\frac{-t}{s}\, e^{-st} - \frac{1}{s^2}\, e^{-st}\right\}(x+y)^{-p} \rangle \tag{5.1}$$

(ii) $$D_y F(s,y) = \langle f(t,x), \left\{\frac{-p}{s}\, e^{-st}\right\}(x+y)^{-p-1} \rangle \tag{5.2}$$

Proof (i):
Let 's' be an arbitrary but fixed point in Ω_f. Choose the positive numbers a_1, b_1 and r such that
$\sigma_1 < a_1 < Re s - r < Re s + r < b_1 < \sigma_2$.
Also, let Δs be a complex increment such that $0 < |\Delta s| < r$.
Consider for $\Delta s \neq 0$, we have

$$\frac{F(s+\Delta s,y)-F(s,y)}{\Delta s} - \langle f(t,x), \frac{\partial}{\partial s}\frac{e^{-st}}{s}(x+y)^{-p} \rangle$$

$$= \frac{1}{\Delta s}\left\{\langle f(t,x), \frac{e^{-(s+\Delta s)t}}{s+\Delta s}(x+y)^{-p} \rangle - \langle f(t,x), \frac{e^{-st}}{s}(x+y)^{-p} \rangle\right\} - \langle f(t,x), \frac{\partial}{\partial s}\frac{e^{-st}}{s}(x+y)^{-p} \rangle$$

$$= \frac{1}{\Delta s}\langle f(t,x), \left\{\frac{e^{-(s+\Delta s)t}}{s+\Delta s} - \frac{e^{-st}}{s}\right\}(x+y)^{-p} \rangle - \langle f(t,x), \frac{\partial}{\partial s}\frac{e^{-st}}{s}(x+y)^{-p} \rangle$$

$$= \quad \langle f(t,x), \frac{1}{\Delta s}\left\{\frac{e^{-(s+\Delta s)t}}{s+\Delta s} - \frac{e^{-st}}{s}\right\} (x+y)^{-p}\rangle - \langle f(t,x), \frac{\partial}{\partial s}\frac{e^{-st}}{s}(x+y)^{-p}\rangle$$

$$= \quad \langle f(t,x), \frac{1}{\Delta s}\left\{\frac{e^{-(s+\Delta s)t}}{s+\Delta s} - \frac{e^{-st}}{s}\right\} (x+y)^{-p} - \frac{\partial}{\partial s}\frac{e^{-st}}{s}(x+y)^{-p}\rangle$$

$$= \quad \langle f(t,x), \varphi_{\Delta S}(t,x)\rangle$$

Where ,

$$\varphi_{\Delta S}(t,x) = \frac{1}{\Delta s}\left\{\frac{e^{-(s+\Delta s)t}}{s+\Delta s} - \frac{e^{-st}}{s}\right\} (x+y)^{-p} - \frac{\partial}{\partial s}\frac{e^{-st}}{s}(x+y)^{-p} \qquad (5.3)$$

To prove that, $\varphi_{\Delta S}(t,x) \in AS_{\alpha}^{\beta}$, we shall show that $|\Delta S| \to 0$,

$\varphi_{\Delta S}(t,x)$ converges in AS_{α}^{β} to zero.

To proceed, let C denotes the circle with centre at 's' and radius r_1 ,

where $0 < r < r_1 < \min(s - a_1, b_1 - s)$.

We may interchange differentiation on s with differentiation on *t* and by using Cauchy's integral formula.

$(-D_t)^l \varphi_{\Delta S}(t,x)$

$$= \frac{1}{\Delta s}(x+y)^{-p}\left[\frac{(-1)^l (s+\Delta s)^l}{s+\Delta s} e^{-(s+\Delta s)t} - \frac{(-1)^l s^l}{s} e^{-st}\right]$$
$$- \frac{\partial}{\partial s}\frac{(-1)^l s^l}{s}\frac{e^{-st}}{s}(x+y)^{-p}$$

$$(D_t)^l \varphi_{\Delta S}(t,x) = \frac{1}{\Delta s}(x+y)^{-p}\left[(s+\Delta s)^{l-1}e^{-(s+\Delta s)t} - s^{l-1} e^{-st}\right] - \frac{\partial}{\partial s}s^{l-1} e^{-st}(x+y)^{-p}$$

Now applying Cauchy's Integral Formula, we get

$$= \frac{1}{\Delta s}(x+y)^{-p}\left[\frac{1}{2\pi i}\int_c \frac{z^{l-1} e^{-zt}}{z-s-\Delta s} dz - \frac{1}{2\pi i}\int_c \frac{z^{l-1} e^{-zt}}{z-s} dz\right] - (x+y)^{-p}\frac{1}{2\pi i}\int_c \frac{z^{l-1} e^{-zt}}{(z-s)^2} dz$$

$$= \frac{1}{\Delta s}(x+y)^{-p}\left[\frac{1}{2\pi i}\int_c \left[\frac{1}{z-s-\Delta s} - \frac{1}{z-s}\right] z^{l-1} e^{-zt} dz\right] - (x+y)^{-p}\frac{1}{2\pi i}\int_c \frac{z^{l-1} e^{-zt}}{(z-s)^2} dz$$

$$= \frac{1}{\Delta s}(x+y)^{-p}\left[\frac{1}{2\pi i}\int_c \left[\frac{z-s-z+s+\Delta s}{(z-s-\Delta s)(z-s)}\right] z^{l-1} e^{-zt} dz\right] - (x+y)^{-p}\frac{1}{2\pi i}\int_c \frac{z^{l-1} e^{-zt}}{(z-s)^2} dz$$

$$= (x+y)^{-p}\left[\frac{1}{2\pi i}\int_c \left[\frac{z^{l-1} e^{-zt}}{(z-s-\Delta s)(z-s)}\right] dz\right] - (x+y)^{-p}\frac{1}{2\pi i}\int_c \frac{z^{l-1} e^{-zt}}{(z-s)^2} dz$$

$$= (x+y)^{-p}\frac{1}{2\pi i}\int_c \left[\frac{1}{(z-s-\Delta s)(z-s)} - \frac{1}{(z-s)^2}\right] z^{l-1} e^{-zt} dz$$

$$= (x+y)^{-p}\frac{1}{2\pi i}\int_c \left[\frac{(z-s)-(z-s-\Delta s)}{(z-s-\Delta s)(z-s)^2}\right] z^{l-1} e^{-zt} dz$$

$$= (x+y)^{-p}\frac{\Delta s}{2\pi i}\int_c \left[\frac{1}{(z-s-\Delta s)(z-s)^2}\right] z^{l-1} e^{-zt} dz$$

$$= \quad (x+y)^{-p}\frac{\Delta s}{2\pi i}\int_c\left[\frac{z^{l-1}\,e^{-zt}}{(z-s-\Delta s)(z-s)^2}\right]dz$$

Now,

$$D_t^l\,(xDx)^q\varphi_{\Delta S}(t,x) = \frac{\Delta s}{2\pi i}p(-p-q)(x+y)^{-p-q}x^q\int_c\left[\frac{z^{l-1}\,e^{-zt}}{(z-s-\Delta s)(z-s)^2}\right]dz$$

$$= \quad \frac{\Delta s\; p(-p-q)(x+y)^{-p-q}x^q}{2\pi i}\int_c\left[\frac{z^{l-1}\,e^{-zt}}{(z-s-\Delta s)(z-s)^2}\right]dz$$

Now, for all $z \in C$ and $-\infty < t < \infty$,

$$\sup_{I_1}\left|K_{a,b}(t)(1+x)^p P(-p-q)(x+y)^{-p-q}x^q\right| \le K$$

where K is constant independent of z and t.

Moreover $|z-s-\Delta s| > r_1 - r > 0$ and $|z-s| = r_1$,

$C_1 = \max\{|z|^{l-1}e^{-zt}, z \in C\}$

Consequently,

$$\sup_{I_1}\left|K_{a,b}(t)(1+x)^p\, D_t^l\,(xDx)^q\varphi_{\Delta S}(t,x)\right|$$

$$\le \quad \sup_{I_1}\left|K_{a,b}(t)(1+x)^p\;\frac{\Delta s\; P(-p-q)(x+y)^{-p-q}x^q}{2\pi i}\int_c\left[\frac{z^{l-1}\,e^{-zt}}{(z-s-\Delta s)(z-s)^2}\right]dz\;\right|$$

$$\le \quad \sup_{I_1}\left|K_{a,b}(t)(1+x)^p P(-p-q)(x+y)^{-p-q}x^q\;\frac{|\Delta s|}{2\pi}\int_c\left[\frac{|z|^{l-1}\,e^{-zt}}{|z-s-\Delta s||z-s|^2}\right]|dz|\;\right|$$

$$\le K\,\frac{|\Delta s|}{2\pi}\int_c\frac{C_1}{(r_1-r)(r_1)^2}\;|dz|$$

$$\le \quad \frac{|\Delta s|}{2\pi}\int_c\frac{KC_1}{(r_1-r)(r_1)^2}\;|dz|$$

$$\le \quad \frac{|\Delta s|}{2\pi}\int_c\frac{C_2}{(r_1-r)(r_1)^2}\;|dz|, \quad \text{where } \; C_2 = KC_1$$

$$\le \quad \frac{|\Delta s|}{2\pi}\;\frac{C_2}{(r_1-r)(r_1)^2}\;2\pi r_1$$

$$\le \quad |\Delta s|\,\frac{C_2}{(r_1-r)(r_1)}$$

The right-hand side is independent of t and converges to zero as $|\Delta s| \to 0$.

This shows that $\varphi_{\Delta S}(t,x)$ converges to zero in AS_α^β as $|\Delta s| \to 0$.

Proof (ii):

Let 'y' be an arbitrary but fixed point. Choose the real positive numbers $a_2, \; b_2 \; and \; h$ such that $\sigma_1 \; < \; a_2 \; < Re\, y - h \; < \; Re\, y + h \; < \; b_2 < \; \sigma_2$.

Also, let Δy be a complex increment such that, $0 < |\Delta y| < h$.

Consider, for $\Delta y \; \neq 0$, we write

$$\frac{F(s,\; y+\Delta y)-F(s,\; y)}{\Delta y} - \left\langle f(t,x), \frac{\partial}{\partial y} \frac{e^{-st}}{s} \, (x+y)^{-p} \right\rangle$$

$$= \frac{1}{\Delta y} \left\{ \left\langle f(t,x), \frac{e^{-st}}{s} \, (x+y+\Delta y)^{-p} \right\rangle - \left\langle f(t,x), \frac{e^{-st}}{s} \, (x+y)^{-p} \right\rangle \right\} - \left\langle f(t,x), \frac{\partial}{\partial y} \frac{e^{-st}}{s} \, (x+y)^{-p} \right\rangle$$

$$= \frac{1}{\Delta y} \left\{ \left\langle f(t,x), \frac{e^{-st}}{s} \, [(x+y+\Delta y)^{-p} - \; (x+y)^{-p} \,] \right\rangle \right\} - \left\langle f(t,x), \frac{\partial}{\partial y} \frac{e^{-st}}{s} \, (x+y)^{-p} \right\rangle$$

$$= \left\langle f(t,x), \frac{1}{\Delta y} \frac{e^{-st}}{s} \, [(x+y+\Delta y)^{-p} - \; (x+y)^{-p}] \right\rangle - \left\langle f(t,x), \frac{\partial}{\partial y} \frac{e^{-st}}{s} \, (x+y)^{-p} \right\rangle$$

$$= \left\langle f(t,x), \frac{1}{\Delta y} \left(\frac{e^{-st}}{s} \right) [(x+y+\Delta y)^{-p} - \; (x+y)^{-p}] - \frac{\partial}{\partial y} \frac{e^{-st}}{s} \, (x+y)^{-p} \right\rangle$$

$$= \langle f(t,x), \varphi_{\Delta y}(t,x) \rangle$$

where,

$$\varphi_{\Delta y}(t,x) = \frac{1}{\Delta y} \left(\frac{e^{-st}}{s} \right) [(x+y+\Delta y)^{-p} - \; (x+y)^{-p}] - \frac{\partial}{\partial y} \frac{e^{-st}}{s} \, (x+y)^{-p} \qquad (5.4)$$

To prove that $\varphi_{\Delta y}(t,x) \; \in \; AS_{\alpha}^{\beta}$.

We shall show that $|\Delta y| \to 0, \;\; \varphi_{\Delta y}(t,x)$ converges in AS_{α}^{β} to zero.

To proceed, let C denotes the circle with centre at y and radius h_1 , where $0 \; < h \; < h_1 < \min\,(y - a_2 \,, \; b_2 - y)$.

We may interchange differentiation on y with differentiation on x and by using Cauchy's Integral Formula,

$$(xD_x)^q \varphi_{\Delta y}(t,x) = \; x^q \frac{1}{\Delta y} \left(\frac{e^{-st}}{s} \right) [P(-p-q)(x+y+\Delta y)^{-p-q} - \; P(-p-q)(x+y)^{-p-q}] - \; x^q \frac{\partial}{\partial y} \frac{e^{-st}}{s} \, P(-p-q)(x+y)^{-p-q}$$

where P(-p-q) is polynomial in (-p-q).

Now by applying Cauchy's Integral Formula,

$$(xD_x)^q \varphi_{\Delta y}(t,x)$$

$$= \; x^q \frac{1}{\Delta y} \left(\frac{e^{-st}}{s} \right) \left\{ \frac{1}{2\pi i} \int_c \frac{P(-p-q)(x+z)^{-p-q}}{z-y-\Delta y} \, dz - \frac{1}{2\pi i} \int_c \frac{P(-p-q)(x+z)^{-p-q}}{z-y} \, dz \right\} - x^q \, \frac{e^{-st}}{s} \, \frac{1}{2\pi i} \int_c \frac{P(-p-q)(x+z)^{-p-q}}{(z-y)^2} \, dz$$

$$= x^q \frac{1}{\Delta y}\left(\frac{e^{-st}}{s}\right)\frac{1}{2\pi i}\int_c \left[\frac{1}{z-(y+\Delta y)} - \frac{1}{z-y}\right] P(-p-q)(x+z)^{-p-q}dz - x^q \frac{e^{-st}}{s}\frac{1}{2\pi i}\int_c \frac{P(-p-q)(x+z)^{-p-q}}{(z-y)^2}\, dz$$

$$= \frac{x^q}{\Delta y}\left(\frac{e^{-st}}{s}\right)\frac{1}{2\pi i}\int_c \left[\frac{z-y-z+y+\Delta y}{(z-y-\Delta y)(z-y)}\right] P(-p-q)(x+z)^{-p-q}dz - x^q \frac{e^{-st}}{s}\frac{1}{2\pi i}\int_c \frac{P(-p-q)(x+z)^{-p-q}}{(z-y)^2}\, dz$$

$$= \frac{x^q}{2\pi i}\left(\frac{e^{-st}}{s}\right)\int_c \left[\frac{1}{(z-y-\Delta y)(z-y)} - \frac{1}{(z-y)^2}\right] P(-p-q)(x+z)^{-p-q}dz$$

$$= \frac{x^q}{2\pi i}\left(\frac{e^{-st}}{s}\right)\int_c \frac{z-y-z+y+\Delta y}{(z-y-\Delta y)(z-y)^2} P(-p-q)(x+z)^{-p-q}dz$$

$$= \frac{x^q \Delta y}{2\pi i}\left(\frac{e^{-st}}{s}\right)\int_c \frac{1}{(z-y-\Delta y)(z-y)^2} P(-p-q)(x+z)^{-p-q}dz$$

Now,

$$(-D_t)^l (xD_x)^q \varphi_{\Delta y}(t,x) = \frac{\Delta y}{2\pi i}(-s)^l\left(\frac{e^{-st}}{s}\right)x^q \int_c \frac{1}{(z-y-\Delta y)(z-y)^2} P(-p-q)(x+z)^{-p-q}dz$$

$$D_t^l (xD_x)^q \varphi_{\Delta y}(t,x) = s^{l-1}e^{-st}\frac{\Delta y}{2\pi i}x^q \int_c \frac{P(-p-q)(x+z)^{-p-q}}{(z-y-\Delta y)(z-y)^2}\, dz$$

Now for all $z \in C$ and $0 < x < \infty$,

$$\sup_{I_1} \left|K_{a,b}(t)(1+x)^p s^{l-1}e^{-st}x^q\right| \le K$$

where K is constant independent of z and x.

Moreover $|z-y-\Delta y| \ge h_1 - h > 0$ and $|z-y| = h_1$

$C_1 = \max\{|P(-p-q)(x+z)^{-p-q}|;\ z \in C\}$

Consequently

$$\sup_{I_1} \left|K_{a,b}(t)(1+x)^p D_t^l (xD_x)^q \varphi_{\Delta y}(t,x)\right|$$

$$= \sup_{I_1} \left|K_{a,b}(t)(1+x)^p \frac{\Delta y\, x^q\, s^{l-1}e^{-st}}{2\pi i}\int_c \frac{P(-p-q)(x+z)^{-p-q}}{(z-y-\Delta y)(z-y)^2}\, dz\right|$$

$$\le \sup_{I_1} \left|K_{a,b}(t)(1+x)^p x^q\, s^{l-1}e^{-st}\right| \frac{|\Delta y|}{2\pi i}\int_c \frac{|P(-p-q)(x+z)^{-p-q}|}{|z-y-\Delta y||z-y|^2}\, |dz|$$

$$\le K\frac{|\Delta y|}{2\pi}\int_c \frac{C_1}{(h_1-h)(h_1)^2}\, |dz|$$

$$\le \frac{|\Delta y|}{2\pi}\int_c \frac{KC_1}{(h_1-h)(h_1)^2}\, |dz|$$

$$\le \frac{|\Delta y|}{2\pi}\int_c \frac{C_2}{(h_1-h)(h_1)^2}\, |dz|$$

Where $C_2 = KC_1$

$$\leq \frac{|\Delta y|}{2\pi} \frac{C_2}{(h_1 - h)(h_1)^2} 2\pi h_1$$

$$\leq |\Delta y| \frac{C_2}{(h_1 - h)(h_1)}$$

The right-hand side is independent of *x* and converges to zero as $|\Delta y| \to 0$.
This shows that $\varphi_{\Delta y}(t, x)$ converges to zero in AS_α^β as $|\Delta y| \to 0$ which ends the proof.

6. Conclusion

To develop the distributional version of the Aboodh–Stieltjes transform rigorously, appropriate testing function spaces have been introduced using the Gelfand–Shilov technique. These spaces provide the necessary mathematical framework for defining the action of distributions and ensure that the transform is well-defined under suitable growth and regularity conditions. The construction of these testing spaces is an important aspect of the present work, as it establishes a systematic connection between the Aboodh–Stieltjes transform and the theory of generalized functions.
In the proposed paper, the distributional Aboodh–Stieltjes transform has been investigated from the viewpoint of Analyticity. By employing the analyticity theorem, it has been established that the transform possesses analytic properties in its appropriate domain under the prescribed assumptions. The testing function spaces play a crucial role in proving this result by providing the required conditions for convergence, continuity, and differentiability of the transformed expression. Consequently, the analyticity result demonstrates that the proposed transform is not merely a formal generalization but constitutes a mathematically consistent integral-transform framework.

7. Authors' Biography

Dr. A. N. Rangari, Professor, Department of Mathematics, Adarsha Science, J. B. Arts and Birla Commerce Mahavidyalaya, Dhamangaon Rly, Dist:- Amravati, 18 Years of Teaching Experience.

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