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Aggregation of Pentagonal Fuzzy Numbers with Fuzzy t-norm

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Abstract

Aggregation operator is one of most crucial processes for combining fuzzy sets that incorporate results of each rule into single fuzzy set. In this paper, a novel category of fuzzy aggregation operators in pentagonal fuzzy numbers (PFNs) was developed. We concentrate in this paper on weighted aggregation operators because we find out the exact numbers while using the uncertainty in a fuzzy environment. And, also the aggregation results are obtained by using the notation of triangular norms as pseudo arithmetic operations (t-norms and t-conorms). Furthermore, we obtain a consequence for pentagonal fuzzy t-norm weighted arithmetic and pentagonal fuzzy t-norm ordered weighted arithmetic. We examine these operator's primary characteristics and compare them to those that exist already. This fuzzy aggregation operator offers novel and intriguing situations in addition to covering a diverse array of beneficial pre-existing fuzzy aggregation operators. Finally, we apply the weighted aggregation procedures that have been shown in the numerical example for the best student performance selection strategies for pentagonal fuzzy numbers.

Keywords: Fuzzy numbers, Pentagonal fuzzy number, Fuzzy triangular norms (t-norms and t-conorms), Fuzzy weighted averaging operators, Fuzzy ordered weighted aggregation operators.

MSC: 03E72.

1. Introduction

Aggregation process is the collection of units or collection of uncertain values that are combined into a single value. OWA (ordered weighted averaging) operator family was invented by Ronald R. Yager, they have allowed us to model linguistically expressed aggregation instruction. It renders parameterized family of mean operators like minimum, maximum, arithmetic average, and median. It is very useful for representing pragmatic approaches to fuzzy environmental problems. There are many authors who recently developed same formulation in wide range of applications [18,20-26]. In real life, there are instances when information can't be evaluated with precise fuzzy numbers, and it is essential to employ an alternative method to describe the data with a high degree of ambiguity.

Merigo and Casanovas [5,6] presented a generalized fuzzy OWA operator along with its use in making strategic decisions as well as another method is probabilistic decision- making using OWA operator which applies the investment management strategy to find out the best result. S. Chen et al. [7, 15] presented a novel approach for managing MCDM (multi-criteria decision-making) problems employing an induced OWA operator. Yuan et al. [18] introduced an FWA (fuzzy weighted average) utilizing a generalized means function. Johnson et al. [9] introduced an aggregation of PFNs with an OWA operator anchored in VIKOR.

This study aims to build a novel class of fuzzy aggregation operators for pentagonal fuzzy numbers utilizing t-norm and t-conorm aggregation operators. This paper is extended to the fuzzy triangular aggregation operators [1] with normalized fuzzy numbers and [28] generalized trapezoidal fuzzy numbers and interval-valued trapezoidal fuzzy numbers with aggregation operator.

The main benefit of this paper, the PFN, is its ability to assess uncertain data in a more comprehensive way by showing both the best and worst cases as well as the possibility of interval values; it also includes a wide range of fuzzy aggregation operators, such as FA (fuzzy average), FWA, and FOWA, using t-norm and t-conorm in pseudo-arithmetic operations.

This work has been structured as given: Section 2 succinctly presents fundamental concepts and definitions. Section 3 delineated new characteristics of aggregation operators in PFNs. Section 4 presents proposed fuzzy aggregation operators in PFNs. A numerical example is shown in Section 5, demonstrating application of new car selection technique. Conclusion of this paper is given in Section 6.

2. Preliminaries

2.1 Fuzzy Numbers: This section covers the FNs, PFNs, GRPFNs, t-norm, t-conorm, arithmetic operations on PFNs, and ranking function of PFNs. Zadeh first proposed the notations for fuzzy numbers. Since then, several authors, including Dubois and Prade [21], have researched and utilized it. The key advantage is that it offers the comprehensive method for determining the exact decision among the possible interval values that may occur.

Definition 1: A fuzzy set $\tilde{A}$ on the real numbers $\Re$, which satisfy at least the following the three properties are called a fuzzy number [12].

(i) $\mu_{\tilde{A}}(x)$ is piecewise continuous.

(ii) $\tilde{A}$ must be a normal fuzzy set.

(iii) $\alpha_{\tilde{A}}$ must be closed interval for every $\alpha \in (0,1]$.

Definition 2: The intersection of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is specified by a function $t: [0,1] \times [0,1] \to [0,1]$, if it is satisfying the following conditions [12]:

1. $t(0,0) = 0\,; t(\mu_{\tilde{A}}(x), 1) = t\big(1, \mu_{\tilde{A}}(x)\big) = \mu_{\tilde{A}}(x), x \epsilon X$. **(Boundary)**
2. $if\ \ \mu_{\tilde{A}}(x) \leq \mu_{\tilde{C}}(x)\ and\ \ \mu_{\tilde{B}}(x) \leq \mu_{\tilde{D}}(x)$ then $t\big(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\big) \leq t\big(\mu_{\tilde{C}}(x), \mu_{\tilde{D}}(x)\big)$. **(Monotonicity)**
3. $t\big(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\big) = t\big(\mu_{\tilde{B}}(x), \mu_{\tilde{A}}(x)\big)$. **(Commutativity)**
4. $t\left(\mu_{\tilde{A}}(x), t\big(\mu_{\tilde{B}}(x), \mu_{\tilde{C}}(x)\big)\right) = t\left(t\big(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\big), \mu_{\tilde{C}}(x)\right)$. **(Associativity)**

Definition 3: A fuzzy set $\tilde{A}_p = (p_1, q_1, r_1, s_1, u_1)$ is defined on the universal set of real numbers $\Re$, is said to be a pentagonal fuzzy number (PFN) if its membership function has the following characteristics [11]:

1. $\mu_{\tilde{A}_p} : \Re \rightarrow [0,1]$ is continuous.
2. $\mu_{\tilde{A}_p}(x) = 0$ for all $x \in (-\infty, p_1] \cup [u_1, \infty)$.
3. $\mu_{\tilde{A}_p}(x)$ is strictly increasing on $[p_1, q_1], [q_1, r_1]$ and strictly decreasing on $[r_1, s_1], [s_1, u_1]$.
4. $\mu_{\tilde{A}_p}(x) = 1$ for all $x \in p_3$, where $p_1 < q_1 < r_1 < s_1 < u_1$.

The membership function for the pentagonal fuzzy number is given by

$$\mu_{\tilde{A}_p}(x) = \begin{cases} 0\ , & x < p_1 \\ \frac{x-p_1}{q_1-p_1}\ , & p_1 \leq x \leq q_1 \\ \frac{x-q_1}{r_1-q_1}\ , & q_1 \leq x \leq r_1 \\ 1\ , & x = r_1 \\ \frac{s_1-x}{s_1-r_1}\ , & r_1 \leq x \leq s_1 \\ \frac{u_1-x}{u_1-s_1}\ , & s_1 \leq x \leq u_1 \\ 0\ , & x > u_1 \end{cases} \quad (1)$$

Definition 4: The GRPFNs $\tilde{A}_p = (m_1, n_1, o_1, p_1, q_1; s)$is a fuzzy subset of real line $\Re$, where m_1, n_1, o_1, p_1, q_1 are real numbers with $0 < s \leq 1$. Then the generalized regular pentagonal fuzzy number and its membership function is defined as [17];

$$\mu_{\tilde{A}_p}(x) = \begin{cases} 0\ ; & x \leq m_1 \\ s\left(\frac{x-m_1}{n_1-m_1}\right) ; & m_1 \leq x \leq n_1 \\ 1-(1-s)\left(\frac{x-n_1}{o_1-n_1}\right) ; & n_1 \leq x \leq o_1 \\ 1\ ; & x = o_1 \\ 1-(1-s)\left(\frac{p_1-x}{p_1-o_1}\right) ; & o_1 \leq x \leq p_1 \\ s\left(\frac{q_1-x}{q_1-p_1}\right) ; & p_1 \leq x \leq q_1 \\ 0\ ; & x \geq q_1 \end{cases} \quad (2)$$

2.2 Arithmetic Operations: We present fundamental arithmetic and ranking function based on PFNs in the interval [0, 1].

2.2.1 Arithmetic operations on PFNs: Assume that there are two PFNs, $\tilde{A}_p = (p_1, q_1, r_1, s_1, u_1)$ and $\tilde{B}_p = (p_2, q_2, r_1, s_2, u_2)$ and the arithmetic operations are defined as [11],

(i) Addition: $\tilde{A}_p + \tilde{B}_p = (p_1 + p_2,\ q_1 + q_2,\ r_1 + r_2,\ s_1 + s_2,\ u_1 + u_2)$.
(ii) Subtraction: $\tilde{A}_p - \tilde{B}_p = (p_1 - p_2,\ q_1 - q_2,\ r_1 - r_2,\ s_1 - s_2,\ u_1 - u_2)$. (3)
(iii)Multiplication: $\tilde{A}_p * \tilde{B}_p = (p_1 * p_2,\ q_1 * q_2,\ r_1 * r_2,\ s_1 * s_2,\ u_1 * u_2)$.

2.2.2 Ranking Function: The ranking function of the generalized regular pentagonal fuzzy number $\tilde{A}_p = (m_1, n_1, o_1, p_1, q_1; s)$ is defined by [17]

$$R(\tilde{A}_p) = \sqrt{\widetilde{x_0^2} + \widetilde{y_0^2}} \quad (4)$$

Where $(\widetilde{x_0}, \widetilde{y_0}) = \left(\frac{{q_1}^2+12\,{p_1}^2+9\,{o_1}^2-2\,{n_1}^2-m_1\,n_1-9\,n_1\,o_1-9\,o_1\,p_1-p_1q_1}{6q_1+18\,p_1-18\,n_1-6m_1}, \frac{s[q_1+11\,p_1-11\,n_1-m_1]}{6q_1+18\,p_1-18\,n_1-6m_1} \right)$

For any two generalized regular pentagonal fuzzy numbers $\tilde{A}_p = (m_1, n_1, o_1, p_1, q_1; s)$ and $\tilde{B}_p = (m_2, n_2, o_2, p_2, q_2; s)$ should satisfies the following three conditions, if

(1) $R(\tilde{A}_p) > R(\tilde{B}_p)$ then $\tilde{A}_p > \tilde{B}_p$.

(2) $R(\tilde{A}_p) < R(\tilde{B}_p)$ then $\tilde{A}_p < \tilde{B}_p$.

(3) $R(\tilde{A}_p) \approx R(\tilde{B}_p)$ then $\tilde{A}_p \approx \tilde{B}_p$.

3 Proposed Method

3.1 Pseudo-arithmetic operations on PFNs: Assume that there are two PFNs, $\tilde{A}_p = (p_1, q_1, r_1, s_1, u_1)$ and $\tilde{B}_p = (p_2, q_2, r_1, s_2, u_2)$, with $0 \leq p_i \leq q_i \leq r_i \leq s_i \leq u_i$ for each $i = 1,2$. That is, $\tilde{A}_p, \tilde{B}_p \in \widetilde{\wp}(\mathfrak{R}^+)$, where $\mathfrak{R}^+$ is a set of fuzzy subsets of positive real numbers and $\widetilde{\wp}$ is the pseudo fuzzy set. The pseudo-arithmetic operations $\{\oplus, \odot\}$ between $\tilde{A}_p$ and $\tilde{B}_p$ are defined as follows [4,15,16,19]:

$$\begin{aligned} &\text{(i) } \tilde{A}_p \oplus \tilde{B}_p = \tilde{B}_p \oplus \tilde{A}_p = (p_1 + p_2,\ q_1 + q_2,\ r_1 + r_2,\ s_1 + s_2,\ u_1 + u_2) \\ &\text{(ii) } \tilde{A}_p \odot \tilde{B}_p = \tilde{B}_p \odot \tilde{A}_p = (p_1.p_2,\ q_1.q_2,\ r_1.r_2,\ s_1.s_2,\ u_1.u_2) \qquad (5) \\ &\text{(iii) } \tilde{A}_p \odot \lambda = \lambda \odot \tilde{A}_p = (\lambda.\tilde{p}_1,\ \lambda.q_1,\ \lambda.r_1,\ \lambda.s_1,\ \lambda.u_1), \text{with } \lambda \geq 0. \end{aligned}$$

3.2 Key Attributes of Fuzzy Aggregation Operators on PFNs: This segment we proposed the characteristics of pentagonal fuzzy aggregation operator inside a fuzzy context. In defining the properties of fuzzy aggregation operators, we utilized the Gödel implication. Fuzzy aggregation operators are often defined as function $\tilde{F} : \widetilde{\wp}(\mathfrak{R})^n \to \widetilde{\wp}(\mathfrak{R})$, with $n \geq 2$ is a natural number. Principal mathematical features of those operators [20] are delineated as given:

1. Boundary conditions: Assume that, for $\tilde{F} : \widetilde{\wp}([0,1])^n \to \widetilde{\wp}([0,1])$ defined as

$$\tilde{F}(0, \ldots, 0) = 0 \text{ and } \tilde{F}(1, \ldots, 1) = 1. \quad (6)$$

2. Monotonicity: For $\tilde{F} : \widetilde{\wp}([0,1])^n \to \widetilde{\wp}([0,1])$ is a monotonicity of a function of i variables defined as

$$\tilde{a}_i \leq \tilde{b}_i \Longrightarrow \tilde{a}_1 \leq \tilde{b}_1, \ldots, \tilde{a}_n \leq \tilde{b}_n \Longrightarrow \tilde{F}(\tilde{a}_1, \ldots, \tilde{a}_n) \leq \tilde{F}(\tilde{b}_1, \ldots, \tilde{b}_n), \forall\, i \in [n]. \quad (7)$$

That is, $\tilde{F}$ is monotonic increasing in all its arguments.

3. Continuity: $\tilde{F}$ is continuous if and only if it is a component-wise continuous operator. That is, $\tilde{F}: \mathfrak{R}^n \to \mathfrak{R}$ is continuous.

Proof: Assume that, for $\tilde{F} : \widetilde{\wp}([0,1])^n \to \widetilde{\wp}([0,1])$ is continuous aggregation operator.

$\Leftrightarrow$ If for all $i \in [0,1]$ the operator $\tilde{F} : \widetilde{\wp}([0,1])^n \to \widetilde{\wp}([0,1])$ is continuous.

$\Leftrightarrow$ That is, if for all $\tilde{a}_1, \ldots, \tilde{a}_n \in [0,1]\ \forall\ (\tilde{a}_{1j})_{j\in\mathfrak{R}}, \ldots, (\tilde{a}_{nj})_{j\in\mathfrak{R}} \in [0,1]^n$: $\lim_{j\to\infty}(\tilde{a}_{ij}) = \tilde{a}_i$.

$\Leftrightarrow$ For $i = 1, \ldots, n$ then $\lim_{j\to\infty} \tilde{F}(\tilde{a}_{1j}, \ldots, \tilde{a}_{nj}) = \tilde{F}(\tilde{a}_1, \ldots, \tilde{a}_n)$.

Hence $\tilde{F}$ is continuous component-wise continuous operator.

4. Commutativity: For $\tilde{F} : \widetilde{\wp}([0,1])^n \to \widetilde{\wp}([0,1])$ is called a commutativity of a function is defined as

$\Rightarrow$ if for all $n \in \mathfrak{R}$, $\forall\ \tilde{a}_1, \ldots, \tilde{a}_n \in [0,1]$

$\Rightarrow \tilde{F}(\tilde{a}_1, \ldots, \tilde{a}_n) = \tilde{F}(\tilde{a}_{\theta(1)}, \ldots, \tilde{a}_{\theta(n)}) \forall\ \theta = (\theta(1), \ldots, \theta(n))$ permutation on $[n]$ (8)

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5. Associativity: For $\tilde{F} : \tilde{\wp}([0,1])^n \to \tilde{\wp}([0,1])$ is called an associativity of a function is defined as

$\Rightarrow$ if for all $n, m, o \in \Re$, for all n > 2, $\forall\ \tilde{a}_1, \dots, \tilde{a}_n, \tilde{b}_1, \dots, \tilde{b}_m, \tilde{c}_1, \dots, \tilde{c}_o \in [0,1]$

$\Rightarrow \tilde{F}(\tilde{a}_1, \dots, \tilde{a}_n, \tilde{b}_1, \dots, \tilde{b}_m, \tilde{c}_1, \dots, \tilde{c}_o) = \tilde{F}\ (\tilde{F}(\tilde{a}_1, \dots, \tilde{a}_n), \tilde{F}(\tilde{b}_1, \dots, \tilde{b}_m), \tilde{F}(\tilde{c}_1, \dots, \tilde{c}_o))$ (9)

$\Rightarrow \tilde{F}((\tilde{a}_1, \dots, \tilde{a}_n), \tilde{F}\ (\tilde{b}_1, \dots, \tilde{b}_m, \tilde{c}_1, \dots, \tilde{c}_o)) = \tilde{F}\ (\tilde{F}(\tilde{a}_1, \dots, \tilde{a}_n, \tilde{b}_1, \dots, \tilde{b}_m), (\tilde{c}_1, \dots, \tilde{c}_o))$.

6. Idempotency: An element $\tilde{u} \in [0,1]$ is called $\tilde{F}$ idempotent element is defined as

$\Rightarrow$ whenever $\tilde{F}\ (\tilde{u}, \dots, \tilde{u}) = \tilde{u}$ for all $n \in \Re^n$.

$\Rightarrow$ That is, $\tilde{F}\ (\tilde{u}, \dots, \tilde{u}) = \tilde{u}$. (10)

7. Bounded: For $\tilde{F} : \tilde{\wp}([0,1])^n \to \tilde{\wp}([0,1])$ is bounded then

(i) t-norm if for every $\tilde{a}_{(n)} \in \tilde{\wp}([0,1])^n$, $\tilde{F}(\tilde{a}_{(n)})$ is bounded by the minimum function.

$$\tilde{F}(\tilde{a}_{(n)}) \le \min(\tilde{a}_{(n)}) = \min(\tilde{a}_1, \dots, \tilde{a}_n).$$

(ii) t-conorm if for every $\tilde{a}_{(n)} \in \tilde{\wp}([0,1])^n$, $\tilde{F}(\tilde{a}_{(n)})$ is bounded by the maximum function.

$$\tilde{F}(\tilde{a}_{(n)}) \le \max(\tilde{a}_{(n)}) = \max(\tilde{a}_1, \dots, \tilde{a}_n).$$

(iii) Average whenever for all $\tilde{a}_{(n)} \in \tilde{\wp}([0,1])^n$ we have,

$$\text{minimum}\ (\tilde{a}_1, \dots, \tilde{a}_n) \le \tilde{F}\ (\tilde{a}_1, \dots, \tilde{a}_n) \le \text{maximum}\ (\tilde{a}_1, \dots, \tilde{a}_n) \quad (11)$$

Remark 1: Note that axioms F1 and F2 are the two essential characteristics of universal fuzzy aggregating operators.

Example: Let us consider three PFNs $\tilde{A}_p = (1,2,3,4,5)$, $\tilde{B}_p = (7,8,9,10,11)$ and $\tilde{C}_p = (0,1,2,3,4)$ To deal with the properties of the above aggregation operator.

Solution:

1. **Boundary conditions:** Let $\tilde{A}_p = (1,2,3,4,5)$, $\tilde{B}_p = (7,8,9,10,11)$ and $\tilde{C}_p = (0,1,2,3,4)$ representing approximate range for PFNs,

$$\mu_{\tilde{A}_p}(a) = \begin{cases} 0, & a < 1 \\ 0.5, & 1 \le a \le 2 \\ 1, & 2 \le a \le 3 \\ 1, & a = 3 \\ 0.5, & 3 \le a \le 4 \\ 0, & 4 \le a \le 5 \\ 0, & a > 5 \end{cases}, \quad \mu_{\tilde{B}_p}(x) = \begin{cases} 0, & b < 7 \\ 0.5, & 7 \le b \le 8 \\ 1, & 8 \le b \le 9 \\ 1, & b = 9 \\ 0.5, & 9 \le b \le 10 \\ 0, & 10 \le b \le 11 \\ 0, & b > 11 \end{cases},$$

$$\mu_{\tilde{C}_p}(x) = \begin{cases} 0, & c < 0 \\ 0.5, & 0 \le c \le 1 \\ 1, & 1 \le c \le 2 \\ 1, & c = 2 \\ 0.5, & 2 \le c \le 3 \\ 0, & 3 \le c \le 4 \\ 0, & c > 4 \end{cases}$$

2. **Monotonicity:** Let $\tilde{A}_p = (1,2,3,4,5)$, $\tilde{B}_p = (7,8,9,10,11)$ and $\tilde{C}_p = (0,1,2,3,4)$ representing approximate ramp on PFNs,

$$\mu_{\tilde{A}_p}(a) = \begin{cases} increasing\ ramp\ on\ [1,2,3], & at\ 0 \leq 0.5 \leq 1 \\ core\ on\ [2,3,4], & at\ \ a = 1 \\ Decresing\ ramp\ on\ [3,4,5], at\ 1 \geq 0.5 \geq 0 \end{cases}$$

$$\mu_{\tilde{B}_p}(x) = \begin{cases} increasing\ ramp\ on\ [7,8,9], & at\ 0 \leq 0.5 \leq 1 \\ core\ on\ [8,9,10], & at\ \ a = 1 \\ Decresing\ ramp\ on\ [9,10,11], at\ 1 \geq 0.5 \geq 0 \end{cases}$$

$$\mu_{\tilde{C}_p}(x) = \begin{cases} increasing\ ramp\ on\ [0,1,2], & at\ 0 \leq 0.5 \leq 1 \\ core\ on\ [1,2,3], & at\ \ a = 1 \\ Decresing\ ramp\ on\ [2,3,4], at\ 1 \geq 0.5 \geq 0 \end{cases}$$

3. **Continuity:** Let $\tilde{A}_p = (1,2,3,4,5)$, $\tilde{B}_p = (7,8,9,10,11)$ and $\tilde{C}_p = (0,1,2,3,4)$ representing continuous for PFNs,

$$\mu_{\tilde{A}_p}(a) = \begin{cases} 0\ , & a < 1 \\ a-1\ , & 1 \leq a \leq 2 \\ a-2\ , & 2 \leq a \leq 3 \\ 1\ \ , & a = 3 \\ 4-a\ \ , & 3 \leq a \leq\ 4 \\ 5-a\ \ , & 4 \leq a \leq\ 5 \\ 0\ \ , & a >\ 5 \end{cases}, \ \mu_{\tilde{B}_p}(x) = \begin{cases} 0\ , & b < 7 \\ b-7\ , & 7 \leq b \leq 8 \\ b-8\ , & 8 \leq b \leq 9 \\ 1\ \ , & b = 9 \\ 10-b\ \ , & 9 \leq b \leq 10 \\ 11-b\ , & 10 \leq b \leq 11 \\ 0\ \ , & b > 11 \end{cases},$$

$$\mu_{\tilde{C}_p}(x) = \begin{cases} 0\ , & c < 0 \\ c-0\ , & 0 \leq c \leq 1 \\ c-1\ , & 1 \leq c \leq 2 \\ 1\ \ , & c = 2 \\ 3-c, & 2 \leq c \leq\ 3 \\ 4-c\ , & 3 \leq c \leq\ 4 \\ 0\ \ , & c >\ 4 \end{cases}$$

4. **Commutativity:** Let $\tilde{A}_p = (1,2,3,4,5)$, $\tilde{B}_p = (7,8,9,10,11)$ representing the addition of PFNs.

Then commutativity $\tilde{A}_p + \tilde{B}_p = \tilde{B}_p + \tilde{A}_p$.

L.H.S $\Rightarrow \tilde{A}_p + \tilde{B}_p = (8,10,12,14,16)$

R.H.S $\Rightarrow \tilde{B}_p + \tilde{A}_p = (8,10,12,14,16)$

$\therefore \tilde{A}_p + \tilde{B}_p = \tilde{B}_p + \tilde{A}_p$.

Let $\tilde{A}_p = (1,2,3,4,5)$, $\tilde{B}_p = (7,8,9,10,11)$ representing the multiplication of PFNs.

Then commutativity $\tilde{A}_p * \tilde{B}_p = \tilde{B}_p * \tilde{A}_p$.

L.H.S $\Rightarrow \tilde{A}_p \widetilde{*B_p} = (7,16,27,40,55)$

R.H.S $\Rightarrow \tilde{B}_p * \tilde{A}_p = (7,16,27,40,55)$

$\therefore \tilde{A}_p * \tilde{B}_p = \tilde{B}_p * \tilde{A}_p$.

Hence $\tilde{A}_p$ and $\tilde{B}_p$ is commutativity.

5. **Associativity:** Let $\tilde{A}_p = (1,2,3,4,5)$, $\tilde{B}_p = (7,8,9,10,11)$ and $\tilde{C}_p = (0,1,2,3,4)$ representing the addition of PFNs.

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Then associativity $(\tilde{A}_p + \tilde{B}_p) + \tilde{C}_p = \tilde{A}_p + (\tilde{B}_p + \tilde{C}_p)$.

$$\text{L.H.S} \Rightarrow (\tilde{A}_p + \tilde{B}_p) + \tilde{C}_p = (8,11,14,17,20)$$
$$\text{R.H.S} \Rightarrow \tilde{A}_p + (\tilde{B}_p + \tilde{C}_p) = (8,11,14,17,20)$$
$$\therefore (\tilde{A}_p + \tilde{B}_p) + \tilde{C}_p = \tilde{A}_p + (\tilde{B}_p + \tilde{C}_p).$$

Let $\tilde{A}_p = (1,2,3,4,5)$, $\tilde{B}_p = (7,8,9,10,11)$ and $\tilde{C}_p = (0,1,2,3,4)$ representing the multiplication of PFNs.

Then associativity $(\tilde{A}_p * \tilde{B}_p) * \tilde{C}_p = \tilde{A}_p * (\tilde{B}_p * \tilde{C}_p)$.

$$\text{L.H.S} \Rightarrow (\tilde{A}_p * \tilde{B}_p) * \tilde{C}_p = (0,16,54,120,220)$$
$$\text{R.H.S} \Rightarrow \tilde{A}_p * (\tilde{B}_p * \tilde{C}_p) = (0,16,54,120,220)$$
$$\therefore (\tilde{A}_p * \tilde{B}_p) * \tilde{C}_p = \tilde{A}_p * (\tilde{B}_p * \tilde{C}_p).$$

Hence $\tilde{A}_p, \tilde{B}_p$ and $\tilde{C}_p$ is commutativity for $n > 2$.

6. **Idempotency:** Let $\tilde{A}_p = (1,2,3,4,5)$, $\tilde{B}_p = (7,8,9,10,11)$ and $\tilde{C}_p = (0,1,2,3,4)$
   (i) max $(\tilde{A}_p, \tilde{A}_p)$=min $(\tilde{A}_p, \tilde{A}_p) = \tilde{A}_p = (1,2,3,4,5)$.
   (ii) max $(\tilde{B}_p, \tilde{B}_p)$=min $(\tilde{B}_p, \tilde{B}_p) = \tilde{B}_p = (7,8,9,10,11)$.
   (iii) max $(\tilde{C}_p, \tilde{C}_p)$=min $(\tilde{C}_p, \tilde{C}_p) = \tilde{C}_p = (0,1,2,3,4)$).

7. **Bounded:** Here $\tilde{A}_p, \tilde{B}_p$and $\tilde{C}_p$ bounded, if $1 \le 2 \le 3 \le 4 \le 5$, $7 \le 8 \le 9 \le 10 \le 11$ and $0 \le 1 \le 2 \le 3 \le 4$.
   (i) The support is $(1,5), (7,11)$ and $(0,4)$ meaning any a, b, c outside this range has a membership grade of 0.
   (ii) For any $a = 3$, $b = 9$ and $c = 2$, the membership grade is exactly 1.
   (iii) Slope:

$$\mu_{\tilde{A}_p}(a) = \begin{cases} left\ ramp\ on\ [1,2,3], at\ 1 \le 0.5 \le 3 \\ right\ ramp\ on\ [3,4,5], at\ 1 \ge 0.5 \ge 0 \end{cases}$$
$$\mu_{\tilde{A}_p}(a) = \begin{cases} left\ ramp\ on\ [7,8,9], at\ 0 \le 0.5 \le 1 \\ right\ ramp\ on\ [9,10,11], at\ 1 \ge 0.5 \ge 0 \end{cases}$$
$$\mu_{\tilde{A}_p}(a) = \begin{cases} left\ ramp\ on\ [0,1,2], at\ 0 \le 0.5 \le 1 \\ right\ ramp\ on\ [2,3,4], at\ 1 \ge 0.5 \ge 0 \end{cases}$$

3.3 Some Fundamental Concept of Fuzzy Aggregation Operators in PFNs

In the following, a vector $s = (s_1, \dots, s_n)$ is called a weighting vector if $s_j \in [0,1], \forall\, j \in [n] = \{1, \dots, n\}$, and $\sum_{j=1}^{n} s_j = 1$. Furthermore, in our demonstration deals with $\tilde{a} = (\tilde{a}_1, \dots, \tilde{a}_n) \in \tilde{\wp}(\Re^+)^n$, $\tilde{a}_j = (m_j, n_j, o_j, p_j, q_j)$.

3.3.1 Fuzzy Weighted Averaging Operator (FWA)

FWA technique was introduced by Dong and Wong [27].
When precise fuzzy numbers cannot be assessed with exact values, alternative methodologies for fuzzy numbers must be utilized.

Definition 5: Fuzzy weighted averaging operator of n dimension is mapping $\tilde{F} : \tilde{\wp}(\Re^{+})^{n} \rightarrow \tilde{\wp}(\Re^{+})$, if there is a weighting vector $s = (s_1, \dots, s_n)$ such that $s_j \in [0,1]$ and $\sum_{j=1}^{n} s_j = 1$ then,

$$\text{FWA}\ (\tilde{a}_1, \dots, \tilde{a}_n) = \oplus_{j\in[n]}(s_j \odot \tilde{a}_j)$$

$$= \left(\sum_{j=1}^{n} s_j \,.\, m_j\,, \sum_{j=1}^{n} s_j \,.\, n_j\,, \sum_{j=1}^{n} s_j \,.\, o_j, \sum_{j=1}^{n} s_j \,.\, p_j, \sum_{j=1}^{n} s_j \,.\, q_j\right)$$

$$= \left(WA(m_1, \dots, m_n), WA\ (n_1, \dots, n_n), WA(o_1\ , \dots, o_n), WA(p_1\ , \dots, p_n), WA(q_1, \dots, q_n)\right). \tag{12}$$

where $\oplus$ and $\odot$ operations are defined by the pseudo arithmetic operations.

Remark 2: It's simple to observe that the WA operator is extended by the FWA operator. If $s_j = s = \frac{1}{n}, \forall\, j \in [n]$ then the FWA is reduced to the FAA operator, and the WA is reduced to the arithmetic averaging operator.

That is, $\text{AA}(\tilde{a}_1, \dots, \tilde{a}_n) = \sum s\,.\,\tilde{a}_j \Rightarrow \text{FAA}\ (\tilde{a}_1, \dots, \tilde{a}_n) = \oplus_{j\in[n]}(s \odot \tilde{a}_j)$.

Remark 3: The standard weighted averaging operators are not associative, but here we are using the basic logical operators, such as strict minimum (min) and strict maximum (max) conditions, which are associative. This ensures that the PFNs satisfy the associativity property. We can state the following results are based on [6].

Theorem 1: The fuzzy weighted averaging operator should satisfy properties of axiom **1, 2, 3, 4, 5** if all $s_j = s$, axiom **6** and **7**.

Proof: Let us consider, $(\tilde{a}_1, \dots, \tilde{a}_n), (\tilde{b}_1, \dots, \tilde{b}_n) \in \tilde{\wp}(\Re^{+})^{n}$ and let $s_j = (s_1, \dots, s_n)$ be a weighting vector.

1. **Boundary conditions:**

$$\begin{aligned} &\text{FWA}\ (0, \dots, 0) = \oplus_{j\in[n]} \left(s_j \odot 0\right) = 0 + \dots + 0 = 0, \\ &\text{FWA}\ (1, \dots, 1) = \oplus_{j\in[n]} \left(s_j \odot 1\right) = \textstyle\sum_{j=1}^{n} s_j = 1. \end{aligned} \tag{13}$$

2. **Monotonicity:** Let $\tilde{a}_j \leq j, \forall\, j \in [n]$.

$$\begin{aligned} &\tilde{a}_j \leq \tilde{b}_j \Longrightarrow s_j \odot \tilde{a}_j \leq s_j \odot \tilde{b}_j \\ &\Longrightarrow \left(s_j \odot \tilde{a}_j\right) \oplus \left(s_i \odot \tilde{a}_i\right) \leq \left(s_j \odot \tilde{b}_j\right) \oplus \left(s_i \odot \tilde{b}_i\right) \\ &\Longrightarrow \oplus_{j\in[n]}(s_j \odot \tilde{a}_j) \leq \oplus_{j\in[n]}(s_j \odot \tilde{b}_j) \\ &\Longrightarrow \text{FWA}\ (\tilde{a}_1, \dots, \tilde{a}_n) \leq \text{FWA}\ \left(\tilde{b}_1, \dots, \tilde{b}_n\right). \end{aligned} \tag{14}$$

3. **Continuity:**
According to Definition 5, if a weighted averaging operator is continuous, then the FWA operato r is also continuous.

4. **Commutativity:** Let $\tilde{F}\left(\tilde{a}_{\theta(1)}, \dots, \tilde{a}_{\theta(n)}\right)$ be a permutation of $(\tilde{a}_1, \dots, \tilde{a}_n)$.
In generally, $\text{FWA}\ (\tilde{a}_1, \dots, \tilde{a}_n) \neq \text{FWA}\left(\tilde{a}_{\theta(1)}, \dots, \tilde{a}_{\theta(n)}\right)$.
However, if $s_j = s = \frac{1}{n}, \forall\, j \in [n]$ then $\text{FWA}\left(\tilde{a}_{\theta(1)}, \dots, \tilde{a}_{\theta(n)}\right) = \oplus_{j\in[n]} \left(s_j \odot \tilde{a}_{\theta(j)}\right)$

$$\begin{aligned} &= \left(s_j \odot \tilde{a}_{\theta(1)}\right) \oplus \left(s_j \odot \tilde{a}_{\theta(2)}\right) \oplus \dots \oplus \left(s_j \odot \tilde{a}_{\theta(n)}\right) \\ &= \left(s_j \odot \tilde{a}_{\theta(n)}\right) \oplus \left(s_j \odot \tilde{a}_{\theta(n-1)}\right) \oplus \dots \oplus \left(s_j \odot \tilde{a}_{\theta(1)}\right) \\ &= \dots = \text{FWA}\ (\tilde{a}_1, \dots, \tilde{a}_n). \end{aligned} \tag{15}$$

5. **Associativity:** Let $\tilde{a}_1, \dots, \tilde{a}_n, \tilde{b}_1, \dots, \tilde{b}_m, \tilde{c}_1, \dots, \tilde{c}_o \in [0,1]$.

If $s_j = s = \frac{1}{n}, \forall\, j \in [n]$ then $\mathrm{FWA}(\tilde{a}_j, \tilde{b}_j, \tilde{c}_j) = \oplus_{j\in[n]} \left(s_j \odot (\tilde{a}_j, \tilde{b}_j, \tilde{c}_j)\right)$

Let $[s_j \odot (\tilde{a}_j \oplus \tilde{b}_j)] \oplus (s_j \odot \tilde{c}_j) = (s_j \odot \tilde{a}_j) \oplus [s_j(\tilde{b}_j \oplus \tilde{c}_j)]$ for all [0,1].

$$[(s_j \odot \tilde{a}_j \oplus s_j \odot \tilde{b}_j)] \oplus (s_j \odot \tilde{c}_j) = (s_j \odot \tilde{a}_j) \oplus [(s_j \odot \tilde{b}_j) \oplus (s_j \odot \tilde{c}_j)] \quad (16)$$

$$\oplus_{j\in[n]} \left(s_j \odot (\tilde{a}_j, \tilde{b}_j, \tilde{c}_j)\right) = \mathrm{FWA}\,(\tilde{a}_1, \dots, \tilde{a}_n, \tilde{b}_1, \dots, \tilde{b}_m, \tilde{c}_1, \dots, \tilde{c}_o).$$

6. **Idempotency:** Assume that $\tilde{a}_j = \tilde{p} = (m_1, n_1, o_1, p_1, q_1)\ \forall\, j \in [n]$.

$$\text{Then FWA}\,(\tilde{p}, \dots, \tilde{p}) = \oplus_{j\in[n]}(s_j \odot \tilde{p})$$
$$= \left(\textstyle\sum_{j=1}^{n} s_j . m_1, \sum_{j=1}^{n} s_j . n_1, \sum_{j=1}^{n} s_j . o_1, \sum_{j=1}^{n} s_j . p_1, \sum_{j=1}^{n} s_j . q_1\right) \quad (17)$$
$$= (m_1, n_1, o_1, p_1, q_1) = \tilde{p}.$$

7. **Bounded:** Suppose minimum $(\tilde{a}_1, \dots, \tilde{a}_n) = \underline{\tilde{a}}$ and maximum $(\tilde{a}_1, \dots, \tilde{a}_n) = \bar{\tilde{a}}$.

Then, $\mathrm{FWA}\,(\tilde{a}_1, \dots, \tilde{a}_n) = \oplus_{j\in[n]}(s_j \odot \tilde{a}_j) \geq \oplus_{j\in[n]}(s_j \odot \underline{\tilde{a}}) \geq \underline{\tilde{a}} \cdot (\sum_{j=1}^{n} s_j) \geq \underline{\tilde{a}}$,

$$\mathrm{FWA}\,(\tilde{a}_1, \dots, \tilde{a}_n) = \oplus_{j\in[n]}(s_j \odot \tilde{a}_j) \leq \oplus_{j\in[n]}(s_j \odot \bar{\tilde{a}}) \leq \bar{\tilde{a}} \cdot (\textstyle\sum_{j=1}^{n} s_j) \leq \bar{\tilde{a}}. \quad (18)$$

$$\therefore \text{minimum}\,(\tilde{a}_1, \dots, \tilde{a}_n) \leq \mathrm{FWA}\,(\tilde{a}_1, \dots, \tilde{a}_n) \leq \text{maximum}\,(\tilde{a}_1, \dots, \tilde{a}_n). \quad (19)$$

3.3.2 Fuzzy Ordered Weighted Averaging Operator

Ronald R.Y. invented a form of operator known as an OWA operator. In many cases where making decisions includes ambiguity and imprecision, the OWA operators are quite helpful. According to S.J. Chen et al. and Ulrich Florian Simo et al. [7,1], this formulation of the OWA operator is generated. In order to determine precise fuzzy numbers, following methods are employed.

Definition 6: FOWA (fuzzy ordered weighted averaging operator) of n dimension is mapping $\tilde{F}: \tilde{\wp}(\Re^+)^n \to \tilde{\wp}(\Re^+)$, if there is a weighting vector $s = (s_1, \dots, s_n)$ such that $s_j \in [0,1]$ and $\sum_{j=1}^{n} s_j = 1$ then,

$$\mathrm{FOWA}\,(\tilde{a}_1, \dots, \tilde{a}_n) = \oplus_{j\in[n]}(s_j \odot \tilde{a}_{(j)}) \quad (20)$$

where $\oplus$ and $\odot$ operations are defined by the pseudo arithmetic operations, $(\cdot)$ is a permutation on [n] such that $\tilde{a}_{(1)} \leq \ldots \leq \tilde{a}_{(n)}$.

Remark 4: The OWA operator is extended by the FOWA operator. Hence

- If s = $(1, 0, \dots, 0)$ then FOWA = $\tilde{a}_{(1)}$ (fuzzy minimum).
- If s = $(0, 0, \dots, 1)$ then FOWA =$\tilde{a}_{(1)}$ (fuzzy maximum).
- If $s_j = s = \frac{1}{n}, \forall\, j \in [n]$, then FOWA = Fuzzy arithmetic averaging operator.
- FOWA = FWA that implies the ranking of $\{\tilde{a}_j, j \in [n]\}$ = the ranking of $\{\tilde{a}_{(j)}, j \in [n]\}$.

Theorem 2: FOWA operator should satisfy properties of axiom **1, 2, 3, 4, 5** if all $s_j = s$, axiom **6** and **7**.

Proof: The properties of axiom **1, 2, 3, 5, 6** and **7** results are similar to Theorem 1.

4. **Commutativity:** Let $\tilde{F}(\tilde{a}_{\theta(1)}, \dots, \tilde{a}_{\theta(n)})$ be a permutation of $(\tilde{a}_1, \dots, \tilde{a}_n)$.

FOWA $(\tilde{a}_1, \dots, \tilde{a}_n) = \oplus_{j\in[n]}(s_j \odot \tilde{a}_{(j)})$, $\tilde{a}_{(j)}$ j$^{\text{th}}$ largest value of $\{\tilde{a}_j, j \in [n]\}$, (21)

$\mathrm{FOWA}(\tilde{a}_{\theta(1)}, \dots, \tilde{a}_{\theta(n)}) = \oplus_{j\in[n]} \left(s \odot \tilde{a}'_{\theta(j)}\right)$, $\tilde{a}'_{\theta(j)}$ j$^{\text{th}}$ largest value of $\{\tilde{a}_{\theta(j)}, j \in [n]\}$.

Since $(\tilde{a}_{\theta(1)}, \dots, \tilde{a}_{\theta(n)})$ is a permutation of $(\tilde{a}_1, \dots, \tilde{a}_n)$, we have $\tilde{a}_{(j)} = \tilde{a}'_{\theta(j)}, \forall\, j \in [n]$.

Hence, FOWA $(\tilde{a}_1, \dots, \tilde{a}_n) = \mathrm{FOWA}(\tilde{a}_{\theta(1)}, \dots, \tilde{a}_{\theta(n)})$. (22)

4. The Fuzzy t- norm Aggregation Operators in PFNs

We provide a novel approach in this section for building fuzzy aggregation operators using t-norms and t-conorms. Fuzzy t-norm aggregation operators in PFNs are name of this new family of aggregation operators that can handle cases when values are to be aggregated. $\tilde{p} = (m, n, o, p, q)$ have been expressed as normalized fuzzy numbers (NFNs). The operational laws on NFNs can be based on t-norms and t-conorms that may be introduced in [14,20,22]. New class of aggregation operators, described as t-norm and t-conorm, is a commonly used set of logical connectives, which are the fuzzy extensions of AND & OR. Instead of in the interval $\{0,1\}$, valuation set is specified in interval $[0,1]$.

Fuzzy Intersections (t-norm) in PFNs: The intersection of two pentagonal fuzzy sets $\tilde{A}_p$ and $\tilde{B}_p$ is specified by a function $T_P: [0,1] \times [0,1] \to [0,1]$ is called an aggregation operator in t-norms such that for all $m_1, n_1, o_1, p_1, q_1 \in [0,1]$ and if it satisfies following properties:

1. $t_p(0,0) = 0\ ; t_p\left(\mu_{\tilde{A}_p}(x), 1\right) = t_p\left(1, \mu_{\tilde{A}_p}(x)\right) = \mu_{\tilde{A}_p}(x), x \in X$ **(Boundary)**
2. if $m_1 \leq o_1,\ o_1 \leq p_1,\ p_1 \leq\ q_1\ and\ \ q_1 \leq n_1$ then $t_p(m_1, n_1) \leq t_p(m_1, o_1) + t_p(\,o_1, p_1) + t_p(\,p_1, q_1) + t_p(\,q_1, n_1)$ for all $m_1, n_1, o_1, p_1,\ \ q_1 \in X$ and for all distinct points $r_1, s_1, u_1 \in X - \{p_1, q_1\}$. [13] **(Monotonicity)**
3. $t_p\left(\mu_{\tilde{A}_p}(x), \mu_{\tilde{B}_p}(x)\right) = t_p\ (\mu_{\tilde{B}_p}(x), \mu_{\tilde{A}_p}(x))$ **(Commutativity)**
4. $t_p\left(\mu_{\tilde{A}_p}(x), t_p\left(\mu_{\tilde{B}_p}(x), \mu_{\tilde{C}_p}(x)\right)\right) = t_p\left(t_p\left(\mu_{\tilde{A}_p}(x), \mu_{\tilde{B}_p}(x)\right), \mu_{\tilde{C}_p}(x)\right)$ **(Associativity)**

Fuzzy Unions (t-conorm) in PFNs: The union of two pentagonal fuzzy sets $\tilde{A}_p$ and $\tilde{B}_p$ is specified by a function $S_P: [0,1] \times [0,1] \to [0,1]$ is called an aggregation operator in t-conorms or s- norm, which as the same properties as t – norm, unless the boundary condition is satisfies as follows,

$$S_P\ (1,1) = 1; S_P\left(\mu_{\tilde{A}_p}, 0\right) = S_P\left(0, \mu_{\tilde{A}_p}\right) = \mu_{\tilde{A}_p}, x \in X.$$

Some Classes of Fuzzy t-norms and t-conorms in PFNs: Existing t-norm and t-conorm operator families are of several significance, both parameterized and non-parameterized. Families of the t-norm that are parameterized are represented by $\odot$ while the families of t-conorm that are non-parameterized are represented by $\oplus$. These sources [5,6,17,27] are utilized to present the example.

1. $\odot_B(\mu_{\tilde{A}_p}(x), \mu_{\tilde{B}_p}(x)) = \text{maximum}(0, \mu_{\tilde{A}_p}(x) + \mu_{\tilde{B}_p}(x) - 1);$
   $\oplus_B(\mu_{\tilde{A}_p}(x), \mu_{\tilde{B}_p}(x)) = \text{minimum}\ (1, \mu_{\tilde{A}_p}(x) + \mu_{\tilde{B}_p}(x))$. (Bounded difference)
2. $\odot_A(\mu_{\tilde{A}_p}(x), \mu_{\tilde{B}_p}(x)) = \ \mu_{\tilde{A}_p}(x) \cdot\ \mu_{\tilde{B}_p}(x);$
   $\oplus_A(\mu_{\tilde{A}_p}(x), \mu_{\tilde{B}_p}(x)) = \mu_{\tilde{A}_p}(x) + \mu_{\tilde{B}_p}(x) - \mu_{\tilde{A}_p}(x) \cdot \mu_{\tilde{B}_p}(x)$. (Algebraic product and sum or Probabilistic theory)
3. $\odot_D(\mu_{\tilde{A}_p}(x), \mu_{\tilde{B}_p}(x)) = \begin{cases} \mu_{\tilde{A}_p}(x),\ if\ \mu_{\tilde{B}_p}(x) = 1 \\ \mu_{\tilde{B}_p}(x),\ if\ \mu_{\tilde{A}_p}(x) = 1 \\ 0,\ \ if\ \mu_{\tilde{A}_p}(x), \mu_{\tilde{B}_p}(x) < 1 \end{cases}$

   $\oplus_D(\mu_{\tilde{A}_p}(x), \mu_{\tilde{B}_p}(x)) = \begin{cases} \mu_{\tilde{A}_p}(x),\ if\ \mu_{\tilde{B}_p}(x) = 0 \\ \mu_{\tilde{B}_p}(x),\ if\ \mu_{\tilde{A}_p}(x) = 0 \\ 1,\ \ if\ \mu_{\tilde{A}_p}(x), \mu_{\tilde{B}_p}(x) > 1 \end{cases}$ (Drastic method)

Definition 7: Let $\tilde{s} = (s_1, s_2, s_3, s_4, s_5)$ and $\tilde{p} = (m, n, o, p, q)$ be two normalized pentagonal fuzzy numbers (NPFNs). Then t-norm and t-conorm are defined as,

$$\tilde{s} \oplus \tilde{p} = (s_1 \oplus m,\ s_2 \oplus n, s_3 \oplus o,\ s_4 \oplus p,\ s_5 \oplus q)$$
$$\tilde{s} \odot \tilde{p} = (s_1 \odot m,\ s_2 \odot n, s_3 \odot o,\ s_4 \odot p,\ s_5 \odot q) \qquad (23)$$
$$\tilde{p}^{\tilde{s}} = (m^{s_1}, n^{s_2},\ o^{s_3}, p^{s_4}, q^{s_5})$$

Definition 8: The weighting vector is associated with t-conorm ($\oplus$) and its defined as, a vector $s = (s_1, \ldots, s_n)$ such that $s_j \in [0,1]$ if and only if the equality (E) is defined by equation.

$$(\mathrm{E}): s_1 \oplus \ldots \oplus s_n = 1 \qquad (24)$$

Remark 5:

- If $\oplus = \oplus_B$, then (E) $\Leftrightarrow$ $\sum_{j=1}^{n} s_j \geq 1$;
- If $\oplus = \oplus_A$, then (E) $\Leftrightarrow \prod_{j=1}^{n}(1 - s_j) = 0$;
- If $\oplus = \oplus_D$, then (E) $\Leftrightarrow \bigvee_{j\in[n]} s_j = 1$.

We directly present normalized pentagonal fuzzy numbers in this work, which is the process of utilizing certain authors' suggested ways to transform these fuzzy weights into the exact fuzzy numbers.

4.1 The Pentagonal Fuzzy t-norm Weighted Arithmetic Operator: When there is uncertainty in the information supplied and other methods, like PFNs, must be used, the pentagonal fuzzy t-norm weighted arithmetic operator is a continuation of the triangular weighted average operator (see Remark 6 below). The primary benefit is in its consideration of both the highest and lowest outcomes that could arise in an uncertain setting, as well as potential interval values are defined as follows;

Definition 9: PFtWA (pentagonal fuzzy t-norm weighted arithmetic) operator is associated with function $PFtWA: \tilde{\wp}([0,1])^n \rightarrow \tilde{\wp}([0,1])$, such that

$$\mathrm{PFtWA}\ (\tilde{a}_1, \ldots, \tilde{a}_n) = \oplus_{j\in[n]}(s_j \odot \tilde{a}_{(j)})$$
$$= \left(\oplus_{j\in[n]} (s_j.m_j), \oplus_{j\in[n]} (s_j.n_j), \oplus_{j\in[n]} (s_j.o_j), \oplus_{j\in[n]} (s_j.p_j), \oplus_{j\in[n]} (s_j.q_j)\right) \qquad (25)$$
$$= \left(tWA(m_1, \ldots, m_n), tWA\ (n_1, \ldots, n_n), tWA(o_1, \ldots, o_n), tWA(p_1, \ldots, p_n), tWA(q_1, \ldots, q_n)\right)$$

where $\oplus$ is a t-conorm, $\odot$ is a t-norm and $(s_1, \ldots, s_n)$ is a weighting vector associated with $\oplus$.

Remark 6: PFtWA (pentagonal fuzzy t-norm weighted arithmetic) operator renders family of aggregating operators with parameters that encompass tWA (t-norm weighted arithmetic) and FWA operators among others.

Proof: When $\tilde{a} = a$, we get PFtWA = tWA.

Let $\oplus \in \{\oplus_B, \oplus_A, \oplus_D\}$ and $\odot \in \{\odot_B, \odot_A, \odot_D\}$, we have

1. $\oplus = \oplus_B$ **and** $\odot = \odot_B$**:**

$$\mathrm{PFtWA}\ (\tilde{a}_1, \ldots, \tilde{a}_n) = \left(\min\left[1, \textstyle\sum_{j=1}^{n}\left(0 \vee (s_j + m_j - 1)\right)\right]\right), \ldots,$$
$$\left(\min\left[1, \textstyle\sum_{j=1}^{n}\left(0 \vee (s_j + q_j - 1)\right)\right]\right) \qquad (26)$$

2. $\oplus = \oplus_B$ **and** $\odot = \odot_A$**:**

$$\mathrm{PFtWA}\ (\tilde{a}_1, \ldots, \tilde{a}_n) = \left(\textstyle\sum_{j=1}^{n} s_j.m_j, \sum_{j=1}^{n} s_j.n_j, \sum_{j=1}^{n} s_j.o_j, \sum_{j=1}^{n} s_j.p_j, \sum_{j=1}^{n} s_j.q_j\right)$$
$$= \left(WA(m_1, \ldots, m_n), WA\ (n_1, \ldots, n_n), WA(o_1, \ldots, o_n), WA(p_1, \ldots, p_n), WA(q_1, \ldots, q_n)\right)$$
$$= \mathrm{FWA}\ (\tilde{a}_1, \ldots, \tilde{a}_n). \qquad (27)$$

3. $\oplus = \oplus_B$ **and** $\odot = \odot_D$**:**

$$\mathrm{PFtWA}\ (\tilde{a}_1, \ldots, \tilde{a}_n) = \begin{pmatrix} \sum_{j=1}^{n} s_j \wedge m_j, \sum_{j=1}^{n} s_j \wedge n_j, \sum_{j=1}^{n} s_j \wedge o_j, \sum_{j=1}^{n} s_j \wedge p_j, \\ \sum_{j=1}^{n} s_j \wedge q_j \end{pmatrix} \qquad (28)$$

4. $\oplus = \oplus_A$ **and** $\odot = \odot_B$**:**

$$\text{PFtWA}\,(\tilde{a}_1,\ldots,\tilde{a}_n) = (1-\textstyle\prod_{j=1}^{n}[1-\left(0 \vee (s_j+m_j-1)\right)],\ldots, \\ 1-\textstyle\prod_{j=1}^{n}[1-\left(0 \vee (s_j+q_j-1)\right)] \tag{29}$$

5. $\oplus = \oplus_A$ **and** $\odot = \odot_A$**:**

$$\text{PFtWA}\,(\tilde{a}_1,\ldots,\tilde{a}_n) = (1-\textstyle\prod_{j=1}^{n}[1-(s_j\cdot m_j)],\ldots,1-\prod_{j=1}^{n}[1-(s_j\cdot q_j)]) \tag{30}$$

6. $\oplus = \oplus_A$ **and** $\odot = \odot_D$**:**

$$\text{PFtWA}\,(\tilde{a}_1,\ldots,\tilde{a}_n) = \left(1-\textstyle\prod_{j=1}^{n}(1-s_j\wedge m_j)\ ,\ldots,1-\prod_{j=1}^{n}(1-s_j\wedge q_j)\right) \tag{31}$$

7. $\oplus = \oplus_D$ **and** $\odot = \odot_B$**:**

$$\text{PFtWA}\,(\tilde{a}_1,\ldots,\tilde{a}_n) = (\vee_{j\in[n]}[0\vee(s_j+m_j-1)],\ldots,\vee_{j\in[n]}[0\vee(s_j+q_j-1)]) \tag{32}$$

8. $\oplus = \oplus_D$ **and** $\odot = \odot_A$**:**

$$\text{PFtWA}\,(\tilde{a}_1,\ldots,\tilde{a}_n) = \begin{pmatrix}\vee_{j\in[n]}(s_j\cdot m_j),\vee_{j\in[n]}\ (s_j\cdot n_i),\vee_{j\in[n]}\ (s_j\cdot o_j),\vee_{j\in[n]}(s_j\cdot p_j),\\ \vee_{j\in[n]}(s_j\cdot q_j)\end{pmatrix} \tag{33}$$

9. $\oplus = \oplus_D$ **and** $\odot = \odot_D$**:**

$$\begin{aligned}\text{PFtWA}(\tilde{a}_1,\ldots,\tilde{a}_n) &= \begin{pmatrix}\vee_{j\in[n]}(s_j\wedge m_j),\vee_{j\in[n]}\ (s_j\wedge n_j),\vee_{j\in[n]}\ (s_j\wedge o_j),\vee_{j\in[n]}(s_j\wedge p_j),\\ \vee_{j\in[n]}(s_j\wedge q_j)\end{pmatrix}\\ &= (W\max(m_1,\ldots,m_n),W\max(n_1,\ldots,n_n),W\max(o_1,\ldots,o_n),\\ &\quad W\max(p_1,\ldots,p_n),W\max\,(q_1,\ldots,q_n))\\ &= FW\,(\tilde{a}_1,\ldots,\tilde{a}_n).\end{aligned} \tag{34}$$

Theorem 3: PFtWA operator satisfies the properties axioms **1, 2, 3, 4, 5** if all $s_j = s$, **6, 7** if and only if $\odot$ is restrictively distributive over $\oplus$.

Proof: Let $(\tilde{a}_1,\ldots,\tilde{a}_n)$, $(\tilde{b}_1,\ldots,\tilde{b}_n)$ be the two pentagonal fuzzy numbers and let $(s_1,\ldots,s_n)$ be a weighting vector (definition 8).

1. Boundary conditions:

$$\text{PFtWA}\,(0,\ldots,0) = \oplus_{j\in[n]}(s_j\odot 0) = 0,$$
$$\text{PFtWA}\,(1,\ldots,1) = \oplus_{j\in[n]}(s_j\odot 1) = 1. \tag{35}$$

2. Monotonicity: Let us assume that $\tilde{a}_j \le \tilde{b}_j, \forall\, j\in[n]$.

$$\begin{aligned}&\tilde{a}_j \le \tilde{b}_j \Longrightarrow s_j\odot\tilde{a}_j \le s_j\odot\tilde{b}_j\\ &\Longrightarrow (s_j\odot\tilde{a}_j)\oplus(s_i\odot\tilde{a}_i) \le (s_j\odot\tilde{b}_j)\oplus(s_i\odot\tilde{b}_i)\\ &\Longrightarrow \oplus_{j\in[n]}(s_j\odot\tilde{a}_j) \le \oplus_{j\in[n]}(s_j\odot\tilde{b}_j)\end{aligned} \tag{36}$$

Therefore, PFtWA $(\tilde{a}_1,\ldots,\tilde{a}_n) \le$PFtWA $(\tilde{b}_1,\ldots,\tilde{b}_n), \forall\, j\in[n]$.

3. Continuity: PFtWA is continuous because it is component-wise continuous.

4. Commutativity: Let $(\tilde{a}_{\theta(1)},\ldots,\tilde{a}_{\theta(n)})$ be a permutation of $(\tilde{a}_1,\ldots,\tilde{a}_n)$.

In generally, PFtWA $(\tilde{a}_1,\ldots,\tilde{a}_n) \neq$ PFtWA$(\tilde{a}_{\theta(1)},\ldots,\tilde{a}_{\theta(n)})$.

However, if $s_j = s = \frac{1}{n}, \forall\, j\in[n]$ then

$$\begin{aligned}\text{FWA}(\tilde{a}_{\theta(1)},\ldots,\tilde{a}_{\theta(n)}) &= \oplus_{j\in[n]}\,(s\odot\tilde{a}_{\theta(j)})\\ &= (s\odot\tilde{a}_{\theta(1)})\oplus(s\odot\tilde{a}_{\theta(2)})\oplus\ldots\oplus(s\odot\tilde{a}_{\theta(n)}) \qquad (37)\\ &= (s\odot\tilde{a}_{\theta(n)})\oplus(s\odot\tilde{a}_{\theta(n-1)})\oplus\ldots\oplus(s\odot\tilde{a}_{\theta(1)})\\ &= \ldots = \text{PFtWA}\,(\tilde{a}_1,\ldots,\tilde{a}_n).\end{aligned}$$

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5. Associativity: Let $\tilde{a}_1,\ldots,\tilde{a}_n,\tilde{b}_1,\ldots,\tilde{b}_m,\tilde{c}_1,\ldots,\tilde{c}_o \in [0,1]$.

If $s_j = s = \frac{1}{n}, \forall\, j \in [n]$ then $\text{FWA}(\tilde{a}_j,\tilde{b}_j,\tilde{c}_j) = \oplus_{j\in[n]}\left(s_j \odot (\tilde{a}_j,\tilde{b}_j,\tilde{c}_j)\right)$

Let $[s_j \odot (\tilde{a}_j \oplus \tilde{b}_j)] \oplus (s_j \odot \tilde{c}_j) = (s_j \odot \tilde{a}_j) \oplus [s_j(\tilde{b}_j \oplus \tilde{c}_j)]$ for all [0,1].

$$[(s_j \odot \tilde{a}_j \oplus s_j \odot \tilde{b}_j)] \oplus (s_j \odot \tilde{c}_j) = (s_j \odot \tilde{a}_j) \oplus [(s_j \odot \tilde{b}_j) \oplus (s_j \odot \tilde{c}_j)] \quad (38)$$

$\oplus_{j\in[n]}\left(s_j \odot (\tilde{a}_j,\tilde{b}_j,\tilde{c}_j)\right) = \text{PFtWA}\,(\tilde{a}_1,\ldots,\tilde{a}_n,\tilde{b}_1,\ldots,\tilde{b}_m,\tilde{c}_1,\ldots,\tilde{c}_o)$.

6. Idempotency: Assume that $\odot$ is restrictively distributive over $\oplus$.

Then
$$\begin{aligned}\text{PFtWA}\,(\tilde{p},\ldots,\tilde{p}) &= \oplus_{j\in[n]}(s_j \odot \tilde{p})\\ &= (s_1 \odot \tilde{p}) \oplus (s_2 \odot \tilde{p}) \oplus \ldots (s_n \odot \tilde{p})\\ &= (s_1 \oplus \ldots, \oplus\, s_n) \odot \tilde{p} = 1 \odot \tilde{p}\\ &= \tilde{p}.\end{aligned}$$

7. Bounded: Assume that $\odot$ is restrictively distributive over $\oplus$.

Suppose minimum $(\tilde{a}_1,\ldots,\tilde{a}_n) = \underline{\tilde{a}}$ and maximum $(\tilde{a}_1,\ldots,\tilde{a}_n) = \bar{\tilde{a}}$. Then the lower case,

$$\underline{\tilde{a}} \le \tilde{a}_j \Longrightarrow s_j \odot \underline{\tilde{a}} \le s_j \odot \tilde{a}_j \quad \text{(i)}$$
$$\underline{\tilde{a}} \le \tilde{a}_i \Longrightarrow s_i \odot \underline{\tilde{a}} \le s_i \odot \tilde{a}_i \quad \text{(ii)}$$

$$\begin{aligned}\text{(i)-(ii)} &\Longrightarrow (s_j \odot \underline{\tilde{a}}) \oplus (s_i \odot \underline{\tilde{a}}) \le (s_j \odot \tilde{a}_j) \oplus (s_i \odot \tilde{a}_i) \quad (39)\\ &\Longrightarrow \oplus_{j\in[n]}(s_j \odot \underline{\tilde{a}}) \le \oplus_{j\in[n]}(s_j \odot \tilde{a}_j)\\ &\Longrightarrow (s_1 \oplus \ldots, \oplus\, s_n) \odot \underline{\tilde{a}} \le \text{PFtWA}\,(\tilde{a}_1,\ldots,\tilde{a}_n)\\ &\Longrightarrow \underline{\tilde{a}} \le \text{PFtWA}\,(\tilde{a}_1,\ldots,\tilde{a}_n).\end{aligned}$$

Similarly, the way to prove that the upper case that is, PFtWA $(\tilde{a}_1,\ldots,\tilde{a}_n) \le \bar{\tilde{a}}$.

Therefore, minimum $(\tilde{a}_1,\ldots,\tilde{a}_n) \le PFtWA\,(\tilde{a}_1,\ldots,\tilde{a}_n) \le$ maximum $(\tilde{a}_1,\ldots,\tilde{a}_n)$ (40)

4.2 The Pentagonal Fuzzy t-norm Ordered Weighted Arithmetic Operator: When there is uncertainty in the information supplied and other methods, like PFNs, must be used, PFtOWA (pentagonal fuzzy t-norm ordered weighted arithmetic operator) is a continuation of triangular ordered weighted average operator (see Remark 7 below) defined as follows;

Definition 10: PFtOWA (pentagonal fuzzy t-norm ordered weighted arithmetic) operator is associated with function PFtOWA: $\wp([0,1])^n \to \wp([0,1])$, such that

$$\text{PFtOWA}\,(\tilde{a}_1,\ldots,\tilde{a}_n) = \oplus_{j\in[n]}(s_j \odot \tilde{a}_{(j)}) \quad (41)$$

where $\oplus$ is a t-conorm, $\odot$ is a t-norm and $(s_1,\ldots,s_n)$ is a weighting vector associated with $\oplus$ and $(\cdot)$ is a permutation on $[n]$ such that $(\tilde{a}_{(1)},\ldots,\tilde{a}_{(n)})$.

Remark 7: PFtOWA operator renders family of aggregating operators with parameters that encompass tWA and FOWA operators among others.

Proof: This resembles the proof discussed in Remark 6.

Theorem 4: PFtOWA operator satisfies the properties of axioms **1, 2, 3, 4, 5** if all $s_j = s$, axiom **6**, **7** if and only if $\odot$ is restrictively distributive over $\oplus$.

Proof: Similar to Theorem 3, axioms 1, 2, 3, 5, 6, and 7 have been completed.

4. Commutativity:

Let $(\tilde{a}_{\theta(1)},\ldots,\tilde{a}_{\theta(n)})$ be a permutation of $(\tilde{a}_1,\ldots,\tilde{a}_n)$.

$$\text{PFtOWA}\ (\tilde{a}_1,\ldots,\tilde{a}_n) = \oplus_{j\in[n]}(s_j\odot\tilde{a}_{(j)}),\ \tilde{a}_{(j)}\ \text{j}^{\text{th}}\ \text{largest value of}\ \{\tilde{a}_j, j\in[n]\}, \tag{42}$$

$$\text{PFtOWA}\big(\tilde{a}_{\theta(1)},\ldots,\tilde{a}_{\theta(n)}\big) = \oplus_{j\in[n]}\big(s\odot\tilde{a}'_{\theta(j)}\big),\ \tilde{a}'_{\theta(j)}\ \text{j}^{\text{th}}\ \text{largest value of}\ \{\tilde{a}_{\theta(j)}, j\in[n]\}.$$

Since $\big(\tilde{a}_{\theta(1)},\ldots,\tilde{a}_{\theta(n)}\big)$ is a permutation of $(\tilde{a}_1,\ldots,\tilde{a}_n)$, we have $\tilde{a}_{(j)} = \tilde{a}'_{\theta(j)}, \forall\, j\in[n]$.

$$\text{PFtOWA}\ (\tilde{a}_1,\ldots,\tilde{a}_n) = \text{PFtOWA}\ \big(\tilde{a}_{\theta(1)},\ldots,\tilde{a}_{\theta(n)}\big). \tag{43}$$

5. Numerical Example: (Strategies for selecting student achievement based on a five-subject academic report) The averaging operator PFtWA is used to evaluate the students' academic achievement based on their final course grades.

Step-1: Let us consider five alternatives and five criteria, such as follows: Topper 1, Topper 2, Topper 3, Topper 4, Topper 5 and Subject 1, Subject 2, Subject 3, Subject 4, Subject 5. Let us suppose that 5 students are evaluated based on their final marks, which are calculated from 0 to 100, in 5 different subjects, and their linguistic scale values are shown in Table 1, where

(i) $\mu \geq 90$ is the highest membership value 1.
(ii) $\mu \geq 60$ is the average membership value 0.5.
(iii)$\mu \geq 30$ is the lowest membership value 0.

Table 1: Student evaluation in linguistic values.

Alternatives	**Subject 1**	**Subject 2**	**Subject 3**	**Subject 4**	**Subject 5**
Topper 1	0.5	0.5	1	1	0.5
Topper 2	0.5	1	1	0.5	0.5
Topper 3	0.5	0.5	0.5	1	0.5
Topper 4	0.5	0.5	0.5	0.5	0.5
Topper 5	1	1	0.5	0.5	0.5

Step-2: Decision expert assigns a weighting vector $s_E = (0.1, 0.2, 0.3, 0.2{,}0.2)$, which represents significance of each expert analysis.

Table 2: Student evaluation by FWAAO.

Alternatives	**Subject 1**	**Subject 2**	**Subject 3**	**Subject 4**	**Subject 5**	**FWAAO**
Topper 1	0.05	0.1	0.3	0.2	0.1	0.75
Topper 2	0.05	0.2	0.3	0.1	0.1	0.75
Topper 3	0.05	0.1	0.15	0.2	0.1	0,6
Topper 4	0.05	0.1	0.15	0.1	0.1	0.5
Topper 5	0.1	0.2	0.15	0.1	0.1	0.65

According to Table 2, Topper 1 and Topper 2 have the equal best performance among all 5 students. Here FWAAO does not provide the best score values directly. Our proposed method is to rectify these issues, and the results are shown in Table 3.

Step-3: Decision expert assigns a weighting vector from criteria-wise $s_C = (0.1{,}0.2{,}0.3{,}0.3{,}0.1)$. Then, we can aggregate results obtained from some particular cases of PFtWA in Remark 6 (see Table 3). Note

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that, s_C indicates the t-conorm is $\oplus_B$. By Table 3, it is easy to see that the system must be able to select the best strategy based on fuzzy overall top performance.

Table 3: Fuzzy Scores Values for each Strategy

	Topper 1	**Topper 2**	**Topper 3**	**Topper 4**	**Topper 5**
$\oplus_B, \odot_B$	0	0	0	0	0
$\oplus_B, \odot_A$	0.06	0.13	0.19	0.14	0.1
$\oplus_B, \odot_D$	0.3	0.6	0.7	0.6	0.5

Step-4: Establishing the ranking options is the last step in the aggregate process. One way to achieve this is to rank strategies based on their overall best performance score. As we can see, the ordering tactics vary based on the aggregation operator and ranking technique employed, and the outcomes may result in different choices. The complete PFNs from Table 3 are shown in Table 4. Calculating the ranking strategies is a simple process. A novel centroid approach for GRPFNs is derived from the ranking order [17] and can be used to select the best one in Table 5.

Table 4: Pentagonal Fuzzy Scores Values for each Strategy

	Topper 1	**Topper 2**	**Topper 3**	**Topper 4**	**Topper 5**
$\oplus_B, \odot_B$	0	0	0	0	0
$\oplus_B, \odot_A$	(0.04,0.05,0.06,0.07,0.08)	(0.11,0.12,0.13,0.14,0.15)	(0.17,0.18,0.19,0.20,0.21)	(0.12,0.13,0.14,0.15,0.16)	(0.08,0.09,0.10,0.11,0.12)
$\oplus_B, \odot_D$	(0.1,0.2,0.3,0.4,0.5)	(0.4,0.5,0.6,0.7,0.8)	(0.5,0.6,0.7,0.8,0.9)	(0.4,0.5,0.6,0.7,0.8)	(0.3,0.4,0.5,0.6,0.7)

Table 5: Ranking Strategies

$\oplus_B, \odot_B$	Topper 1= Topper 2= Topper 3= Topper 4= Topper 5
$\oplus_B, \odot_A$	Topper 5< Topper 2< Topper 4< Topper 3< Topper 1
$\oplus_B, \odot_D$	Topper 1< Topper 5< Topper 4= Topper 2< Topper 3

6. Conclusion:

We have presented a novel category of pentagonal aggregation operators with t-norm. Aggregation operator employs ambiguous information as PFNs. It is particularly helpful in scenarios with fuzzy environments where decision-makers can evaluate information using FNs but cannot evaluate it using precise numbers or singletons. Their construction method depends on t-norm and t-conorm, pseudo arithmetic operations, and we are giving other specialties of three classes of fuzzy t-norm and t-conorms, such as bounded difference or Lucasiewicz, algebraic product or probabilistic, and Drastic method. Furthermore, we get a result for PFtWA and PFtOWA. We have proved the multi-criteria decision-making problem concerning the best student performance selection strategies based on an overall academic report, utilizing a specific aggregation operator, with results potentially varying among decision-makers. The main advantages of this paper are the weighted average of the aggregation operator, which deals with the many wide-range situations and fuzzy risk analysis.

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Conflicts of Interest:

The authors declare no conflicts of interest.

References:

1. Ulrich Florian Simo and Henri Gwet, Fuzzy Triangular Aggregation Operators, International journal of Mathematics and Mathematical Sciences, Hindawi, Volume 2018, https://doi.org/10.1155/2018/9209524.
2. Apurba Panda and Madhumangal Pal, A study on pentagonal fuzzy number and its corresponding matrices, Pacific Science Review B: Humanities and Social Sciences 1 (2015) 131-139.
3. V. Vidhya, K. Ganesan, A New Ranking Approach for Solving Fuzzy Transportation Problem with Pentagonal Fuzzy Numbers, Mathematics and Statistics 10(4): 816-824, 2022, DOI: 10.13189/ms.2022.100412.
4. N. Sarala and S. Jothi, Pseudo Arithmetic Operations on Fuzzy Measure, International Journal of Mathematics Trends and Technology (IJMTT), Volume 57, Issue 2, May 2018.
5. Jose M. Merigo, Probabilistic Decision Making with the OWA Operator and its Application in Investment Management, IFSA-EUSFLAT 2009.
6. Jose M. Merigo and Montserrat Casanovas, The Fuzzy Generalized OWA Operator and its Application in Strategic Decision Making, Cybernetics and Systems: An International Journal, 41:359-370, DOI: 10.1080/0196722.2010.486223.
7. Shi-Jay Chen and Shyi-Ming Chen, A new method for handling multi-criteria fuzzy decision-making problems using FN-IOWA operators, Cybernetics and Systems: An International Journal, 34: 109-137,2003, DOI: 10.1080/01969720390180230.
8. Intan Arfina and Mashadi, Alternative Arithmetic of Pentagonal Fuzzy Numbers, International Journal of Mathematics Trends and Technology, Volume 66, Issue 12,28-36, 2020, DOI: 10.14445/22315373.
9. S. Johnson Savarimuthu and T. Pathinathan, Aggregation of Pentagonal Fuzzy Numbers with Ordered Weighted Averaging Operator based VIKOR, International Journal of Pure and Applied Mathematics, Volume 119, No. 9 (2018), 295-311.
10. George Biswas, Totan Garai and Uttaran Santra, A possibility-based multi-criteria decision-making approach for artificial recharge structure selection using pentagonal fuzzy numbers, Decision Analytics Journal 9 (2023) 100365.
11. T. Pathinathan, K. Ponnivalavan, Pentagonal Fuzzy Number, International Journal of Computing Algorithm, Volume: 03, May 2014, Pages: 1003-1005.
12. George J. Klir and Boyan, Fuzzy sets and fuzzy logic theory and applications, pearson 2018.
13. Abba Auwalu and Evren Hincal, Common Fixed Points of Two Maps in Cone Pentagonal Metric Spaces, Global Journal of Pure and Applied Mathematics, Volume 12, Number 3 (2016), pp. 2423-2435.
14. Swati Balkrishna Kurdekar, A Study of Fuzzy Metric Space, IJEDR, Volume 7, Issue 3 (2019).

15. S. J. Chen and C.L. Hwang, Fuzzy Multiple Attribute Decision Making: Methods and Applications, vol.375 of Lecture Notes in Economics and Mathematical Systems, Springer, New York, NY, USA, 1992.
16. A. Kaufmann and M. M. Gupta, Introduction to fuzzy arithmetic, Van Nostrand Reinhold Electrical/ Computer Science and Engineering Series, Van Nostrand Reinhold Co., New York, 1985.
17. Swetha T S and Dr. B. Abirami, A New Centroids Method for Ranking of Generalized Regular Pentagonal Fuzzy Numbers, Journal of Nonlinear Analysis and Optimization, Volume 15, Issue 1 (2024).
18. Yuh-Yuan Guh, Rung-Wei Po and E. Stanley Lee, The Fuzzy Weighted Average within a Generalized Means Function, Computers and Mathematics with Applications, ELSEVIER, 55 (2008), pp 2699-2706.
19. Sukanta Nayak and Snehashish Chakraverty, Pseudo Fuzzy Set, Annals of Fuzzy Mathematics and Informatics, Volume 14, No.3, (Sep 2017), pp. 237-248.
20. Tomasa Calvo, Anna kolesarova, Magda Komornikova and Radko mesiar, Aggregation Operators: Properties, Classes and Construction Methods, Aggregation operators New Trends Applications, https://www.researchgate.net/publication/285874074, Jan 2002.
21. D. Dubois and H. Prade, A review of fuzzy set aggregation connectives, Information Sciences, vol.36, no. 1-2, pp.85–121, 1985.
22. B. Schweizer and A. Sklar, Statistical metric spaces, Pacific Journal of Mathematics, vol.10, pp.313–334,1960.
23. H. J. Zimmerman, Fuzzy Set Theory and its Applications, Kluwer Academic Publishers, IInd and IVth Revised Edition.
24. Jose M. Merigo, OWA operators in the weighted average and their application in decision making, Control and Cybernetics, vol. 41 (2012) No. 3.
25. Shouzhen Zeng, Weihua Su, and Anbo Le, Fuzzy Generalized Ordered Weighted Averaging Distance Operator and Its Application to Decision Making, International Journal of Fuzzy Systems, Vol. 14, No. 3, September 2012.
26. Robert Fuller, OWA Operators in Decision Making, August 1996, Research Gate, https://www.researchgate.net/publication/228553904.
27. W. M. Dong and F. S. Wong, Fuzzy Weighted Averages and Implementation of the Extension Principle, Fuzzy Sets and Systems, vol. 21, no. 2, pp.183-199, 1987.
28. Peide Liu, A Weighted Aggregation Operators Multi-Attribute Group Decision Making Method Based on Interval-Valued Trapezoidal Fuzzy Numbers, Expert Systems with Applications, 38 (2011).
29. Azadeh Zahedi Khameneh and Adem Kilicman, Some Construction Methods of Aggregation Operators in Decision-Making Problems: An Overview, Symmetry 2020, 12, 694; doi:10.3390/sym12050694